Some Closure Results for Polynomial Factorization and Applications

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Joint work with Chi-Ning Chou (Harvard) and Noam Solomon (MIT)

Multivariate Polynomials

$$P = \sum_{\mathbf{e}} \alpha_{\mathbf{e}} \mathbf{X}^{\mathbf{e}}$$

$$\mathbf{e} = (e_1, e_2, \ldots, e_n), \sum_i e_i \leq d \qquad \mathbf{X}^{\mathbf{e}} = X_1^{e_1} X_2^{e_2} \cdots X_n^{e_n}$$

$\alpha_{\mathbf{e}}$- field elements

Polynomial on n variables, of degree d.

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- Boolean Circuit Complexity (Razborov-Smolensky)
- Polynomial Method in Combinatorics (Kakeya sets, Distinct distances, Joints problem, Cap sets)

Computation with Multivariate Polynomials

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But wait, how is P given ?

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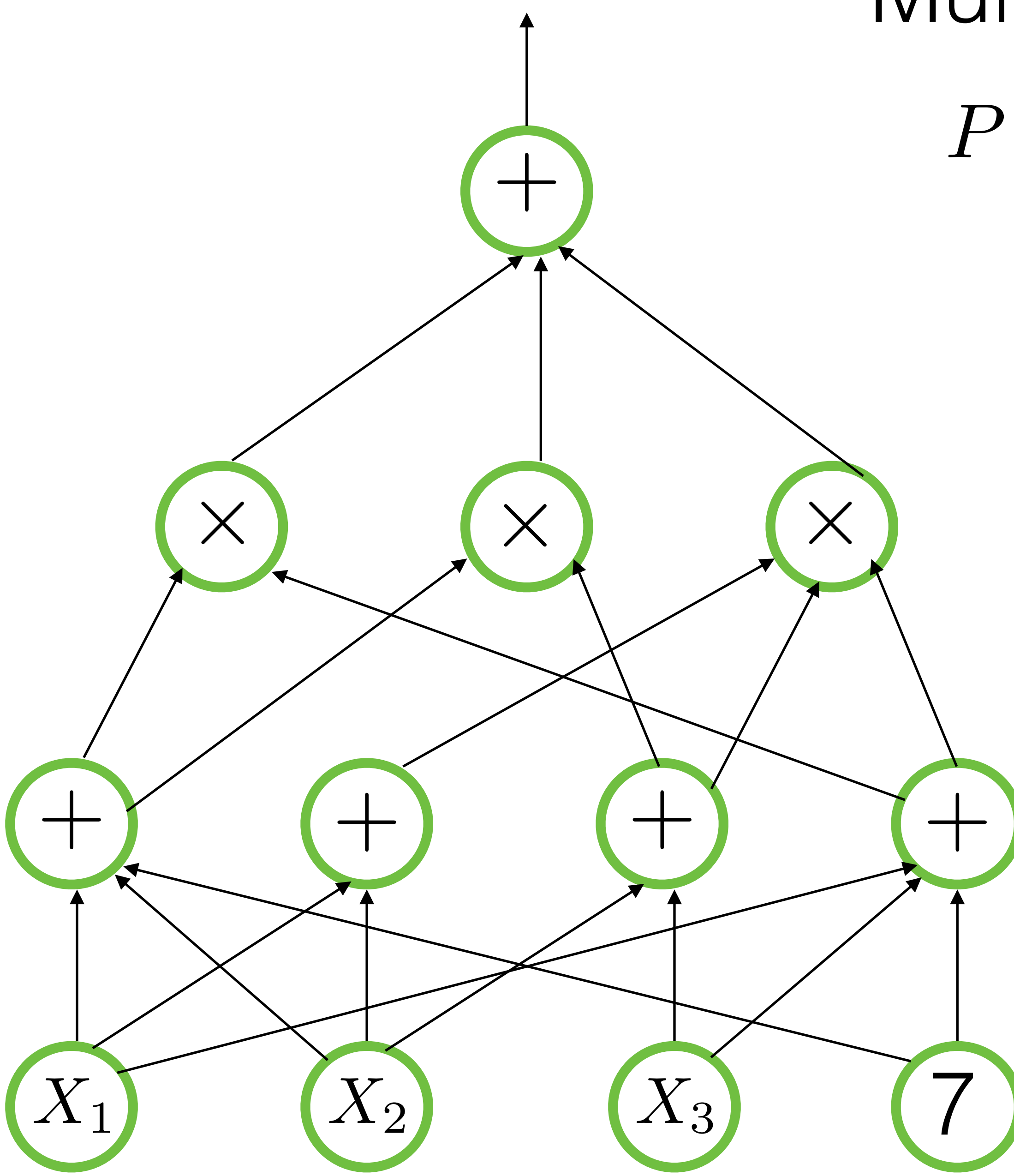
Is there a representation which is more succinct than sum of monomials?

Arithmetic circuits



Multivariate polynomial

$$P \in \mathbb{F}[X_1, X_2, \ldots, X_n]$$

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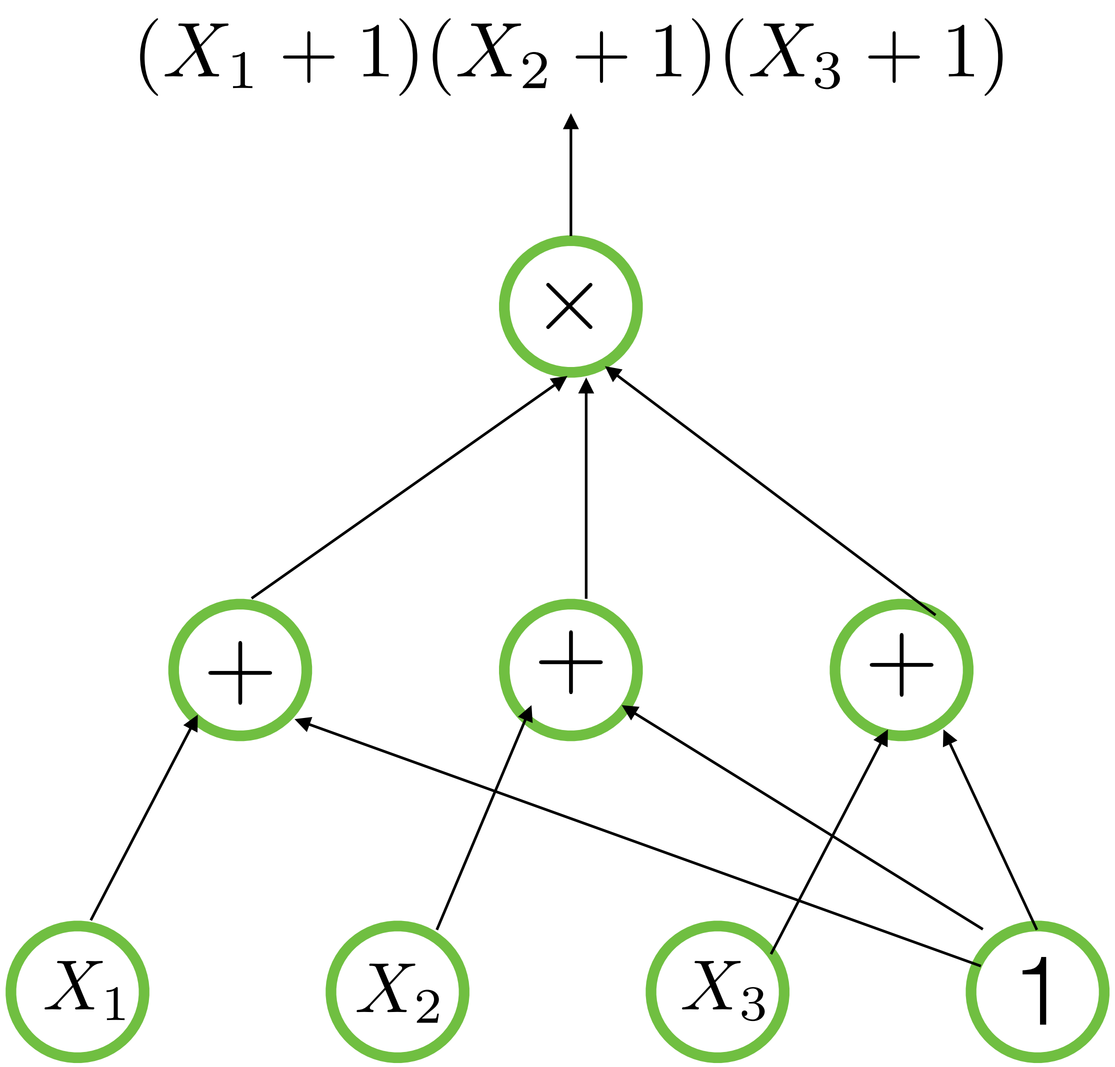

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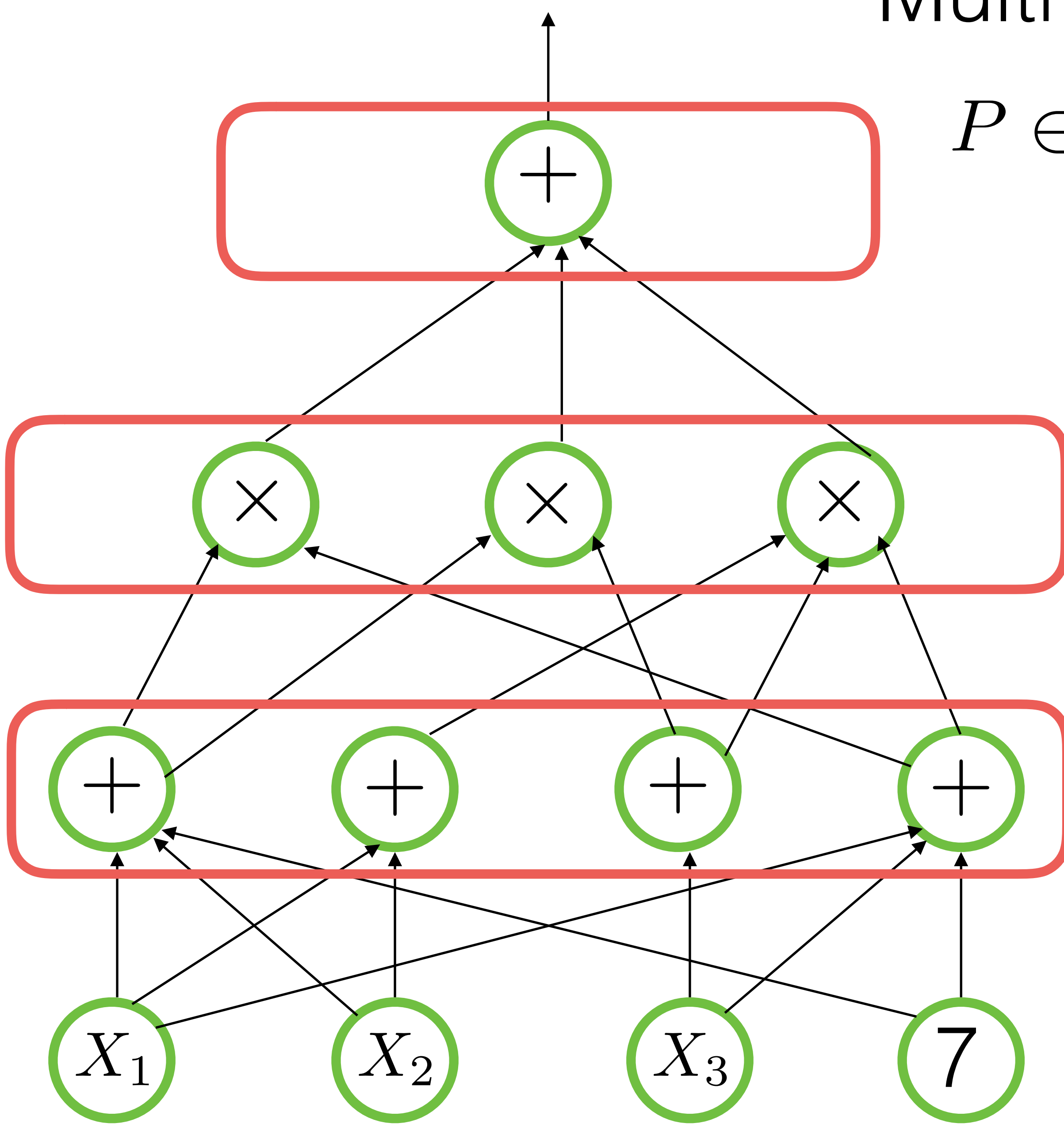

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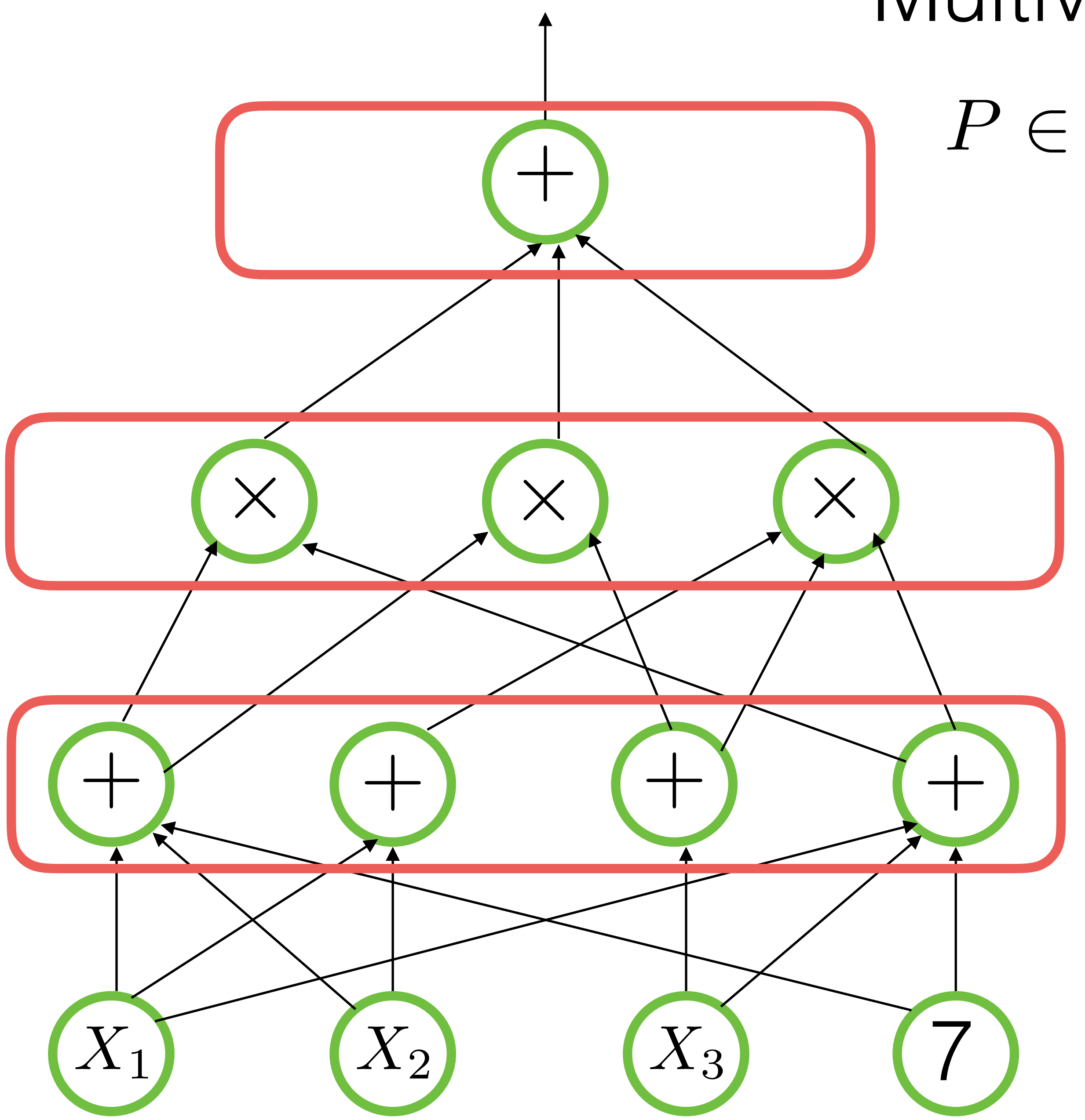
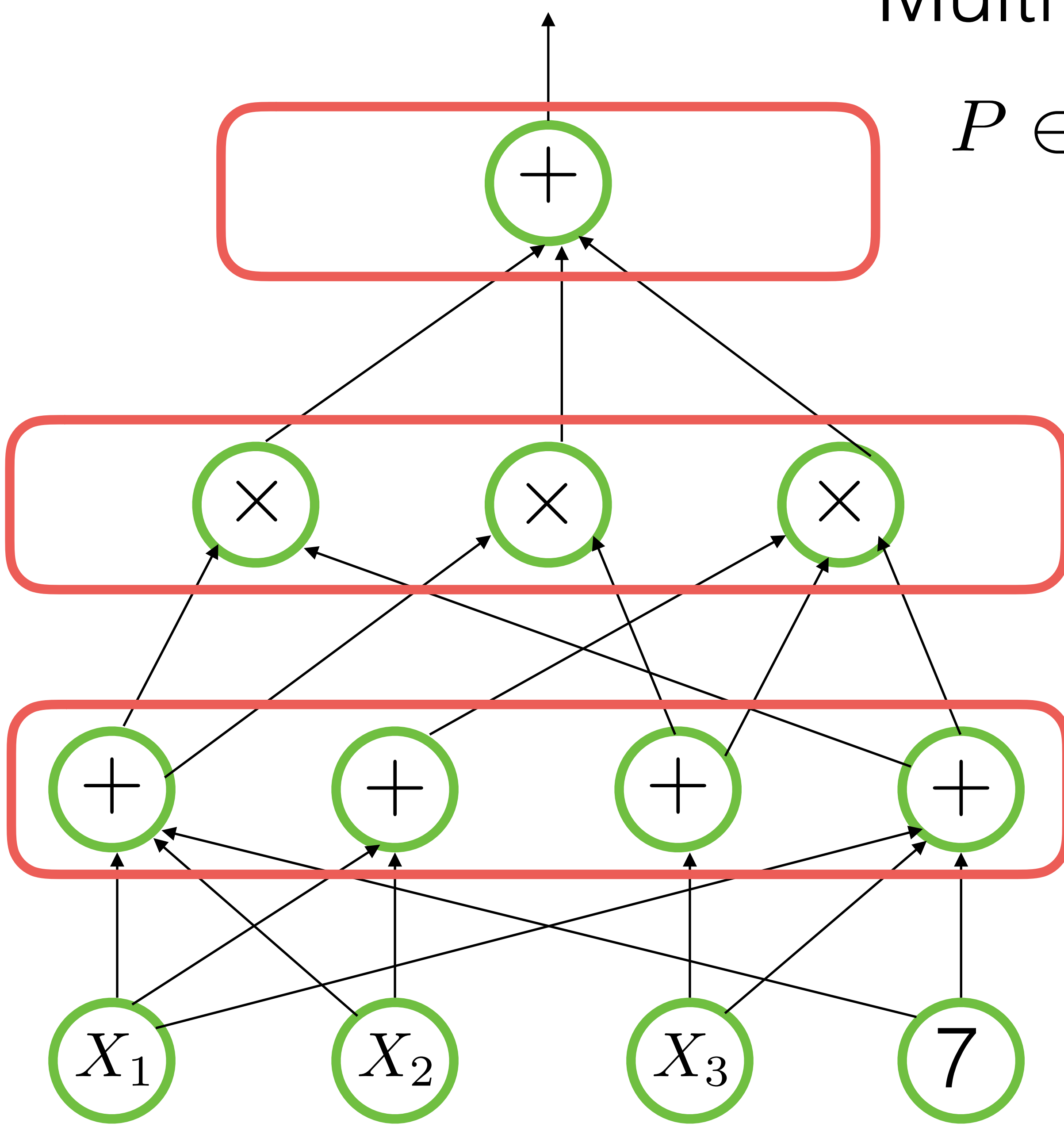


Depth - 3

Size - 8

Arithmetic circuits



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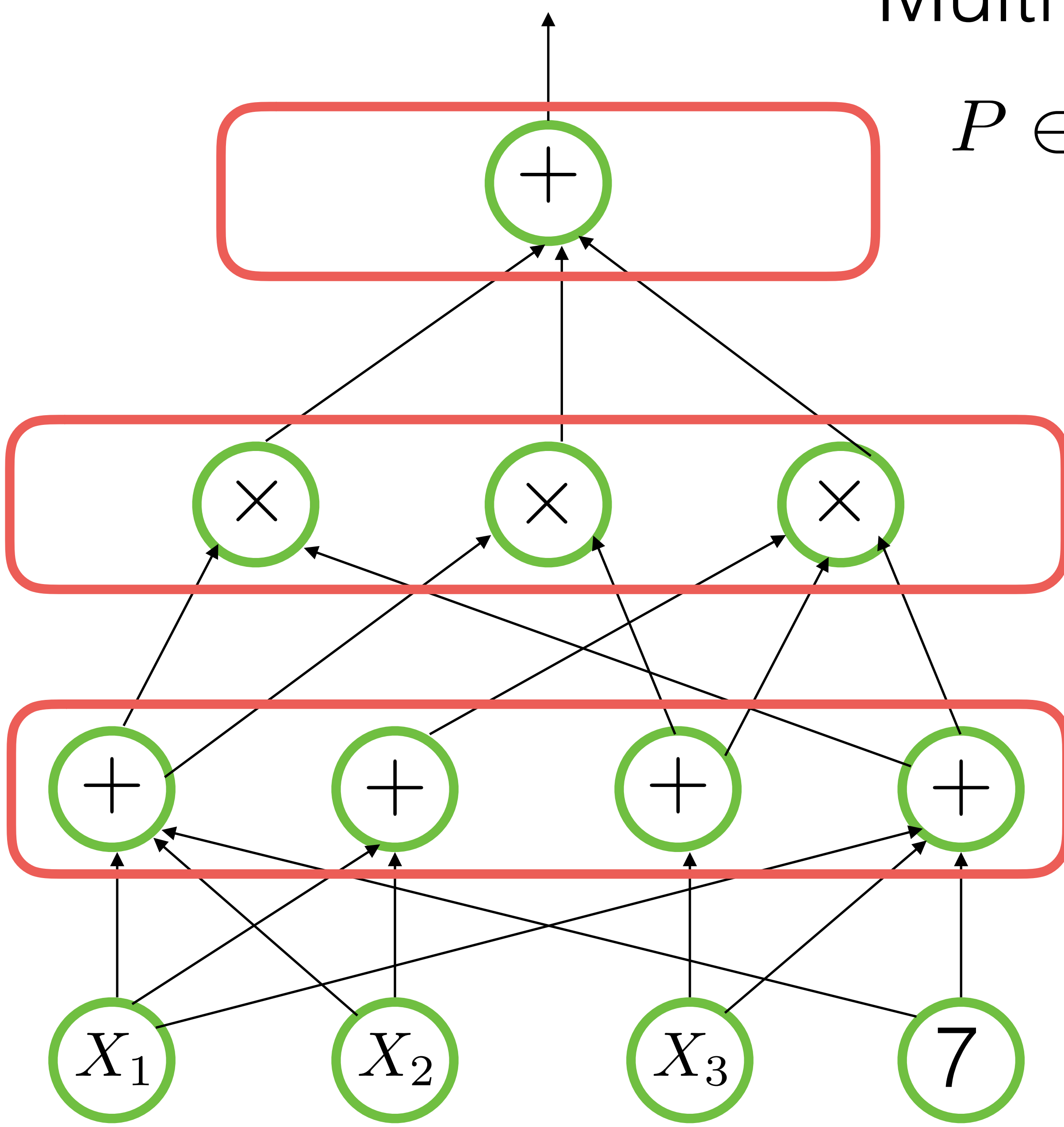

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A circuit is called a formula if the underlying graph is a tree.

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But, can we compute their factors efficiently ?

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Not true for sparse representation!

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Another reason why this representation is not so nice…

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The complexity class VP is uniformly closed under taking factors.

Detour : Algebraic P and NP

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[Folklore, Ben-Or] Esym(m,d) is in VP.

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[Valiant] Permanent is complete for VNP.

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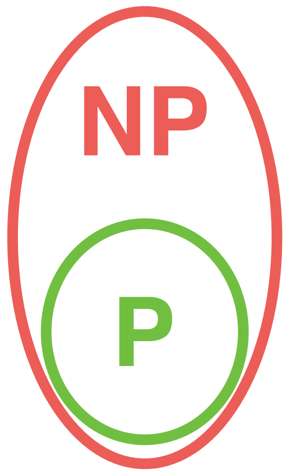
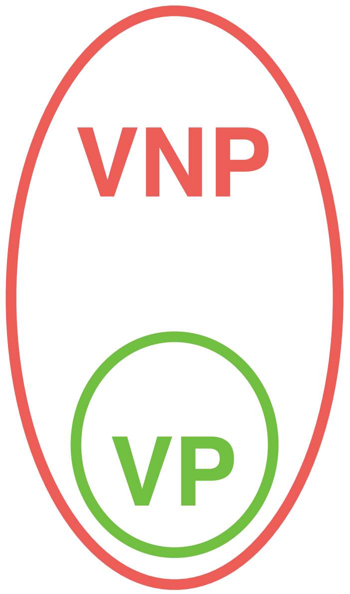
- In particular, he conjectured that Permanent does not have poly(m) sized arithmetic circuits.
- Algebraic analogue of the P vs NP question.

Cook's vs Valiant's hypothesis

P vs NP



VP vs VNP



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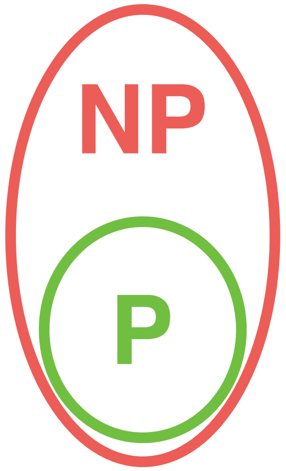

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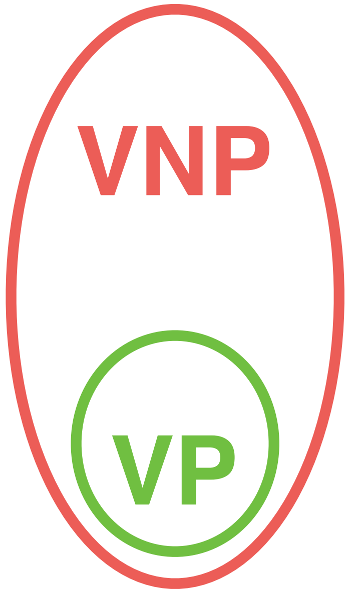

[Burgisser] Under GRH, VP = VNP implies non-uniform P = non-uniform NP.

Algebraic P vs Algebraic NP

Are there explicit polynomial families which cannot be computed by polynomial sized arithmetic circuits ?

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- Applications to derandomization.

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The complexity class VP is uniformly closed under taking factors!

What about closure of other classes ?

- If a polynomial is in VNP, are the factors in VNP ?

- If a polynomial has small formulas, do its factors have small formulas ?

- If a polynomial has small constant depth circuits, do the factors have small constant depth circuits ?

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Conjecture [Burgisser]

The complexity class VNP is closed under taking factors.

Results I : Closure results

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Improves a quasi-polynomial upper bound of Dutta-Saxena-Sinhababu.

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Theorem [Chou-K-Solomon, Dutta-Saxena-Sinhababu]

Let P be an n-variate degree D polynomial computable by a formula of size s, and let f be a factor of degree d of P. Then, f can be computed by a formula of size $\mathbf{d}^{\mathbf{O}(\log \mathbf{d})} \cdot \mathrm{poly}(\mathbf{n}, \mathbf{s}, \mathbf{D})$

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A bound of $\mathbf{n}^{\mathbf{O}(\mathbf{d}^{\epsilon})} \cdot \mathrm{poly}(\mathbf{n}, \mathbf{s}, \mathbf{D})$ **follows from Kaltofen's result and standard structure theorems, but this is not poly(n,s) as long as d is growing.**

Results II : Applications to Hardness vs Randomness

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Two fundamental family of questions in computational complexity

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Derandomization : does every problem with an efficient randomized algorithm have an efficient deterministic algorithm

Hardness vs Randomness

Two fundamental family of questions in computational complexity

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A natural question on its own, but some unexpected and remarkable connections to lower bounds and algorithm design.

A simple randomized algorithm

Lemma [Ore, Schwartz, Zippel, DeMillo, Lipton]

Let S be a subset of the field. Then,

$$\Pr_{\mathbf{a}\in\mathbf{S}^{\mathbf{n}}}[\mathbf{C}(\mathbf{a}) = \mathbf{0}] \leq \frac{\mathbf{d}}{|\mathbf{S}|}$$

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Also, gives an *exp(n log d)* time deterministic algorithm. We are interested in doing anything better than this!

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Explicit Lower Bounds for Arithmetic Circuits

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Theorem [Kabanets-Impagliazzo]

Super-polynomial lower bounds for arithmetic circuits imply non-trivial deterministic PIT for polynomial size arithmetic circuits.

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Crucially, this proof uses Kaltofen's result about closure of VP under factorization.

And thus, does not extend to formulas or low depth circuits, where we do not know closure results.

Scaled down versions of this result ?

Randomness from hardness at low depth

Question [Shpilka-Yehudayoff]

Do lower bounds for low depth circuits imply deterministic PIT for them ?

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Theorem [Dvir-Shpilka-Yehudayoff]

Lower bounds for low depth circuits imply deterministic PIT for low depth circuits with bounded individual degree.

Randomness from hardness at low depth

Theorem [Chou-K-Solomon]

Super-polynomial lower bounds for low depth arithmetic circuits for poly(log n) degree polynomials imply non-trivial deterministic PIT for them.

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For depth k PIT, we need lower bounds for depth k+5 circuits, which as of now, renders this result unusable.

Randomness from hardness for formulas

Theorem [Chou-K-Solomon]

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So, (seemingly) small improvement in the state of lower bounds for formulas has extremely interesting consequences for the PIT question.

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- Low (but growing) degree factors of small formulas, low depth circuits have small formulas, low depth circuits respectively.
- Even somewhat non-trivial lower bounds for formulas, low depth circuits imply sub exponential time deterministic Identity Testing algorithms for them.

Snippets of the proof

Key lemma : structure of factors

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Lemma (informal)

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'Normal form' for factors.

Would be helpful in arguing about their structure.

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P, n-variate, degree D size s circuit.

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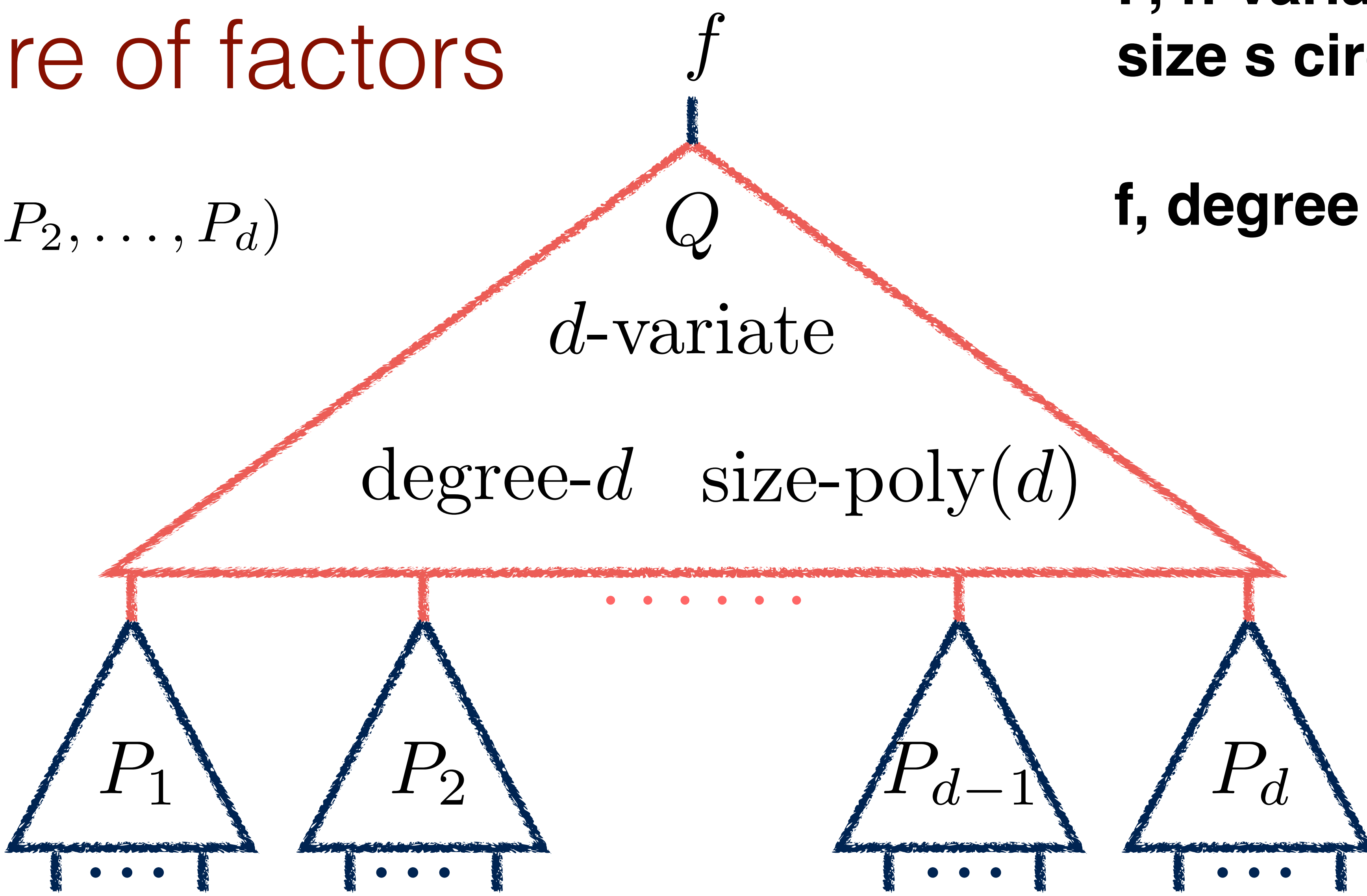

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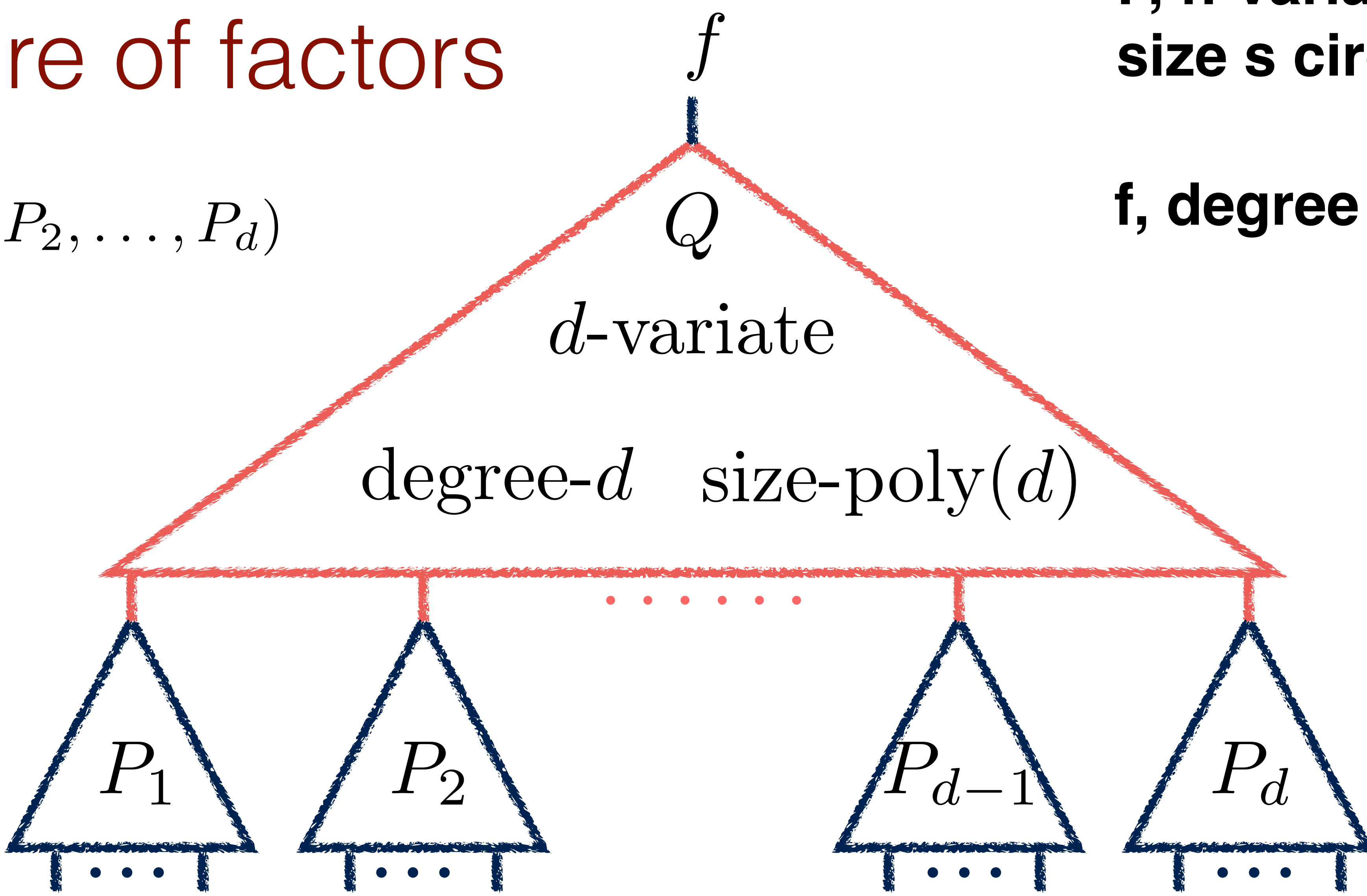

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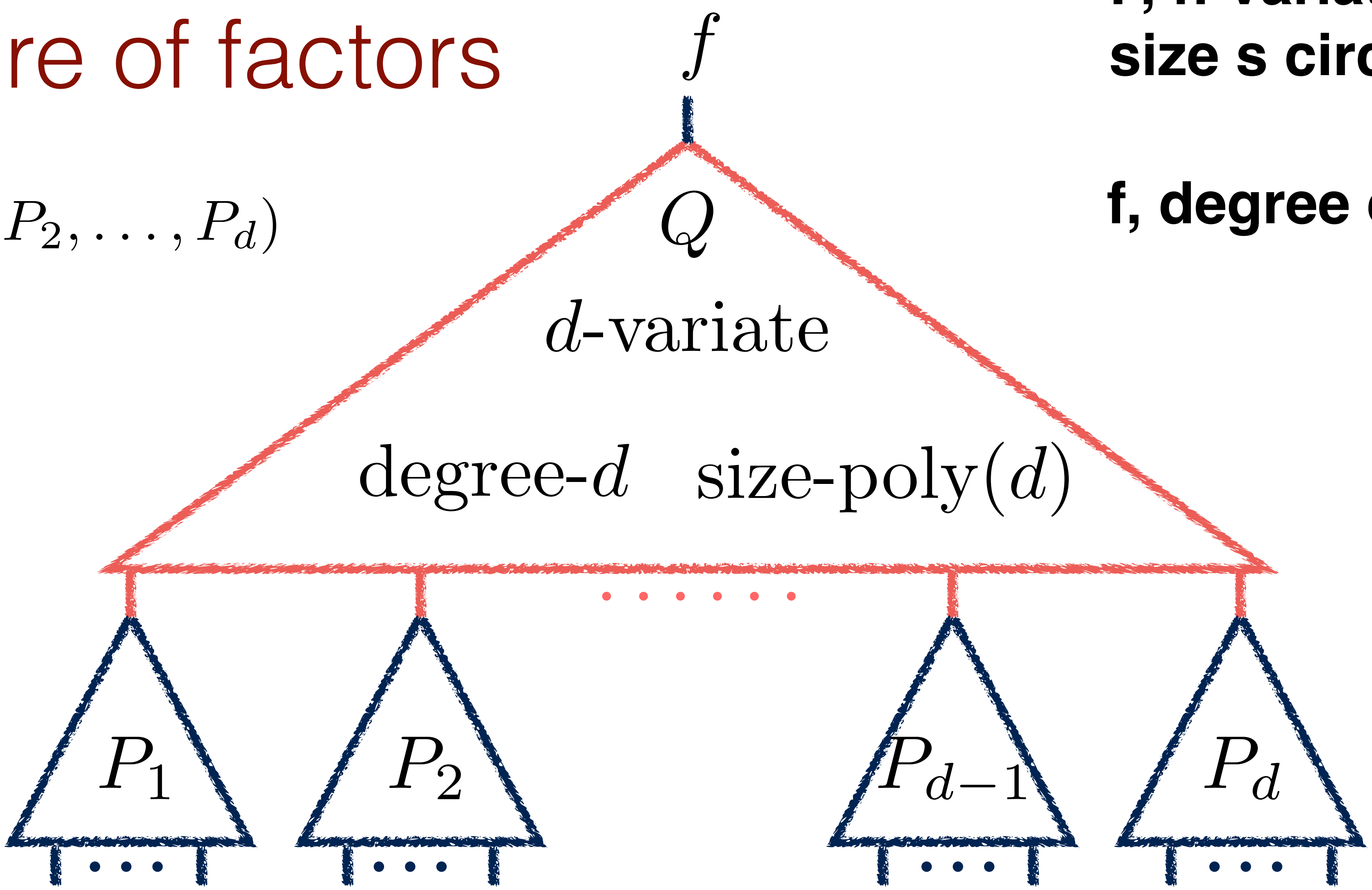

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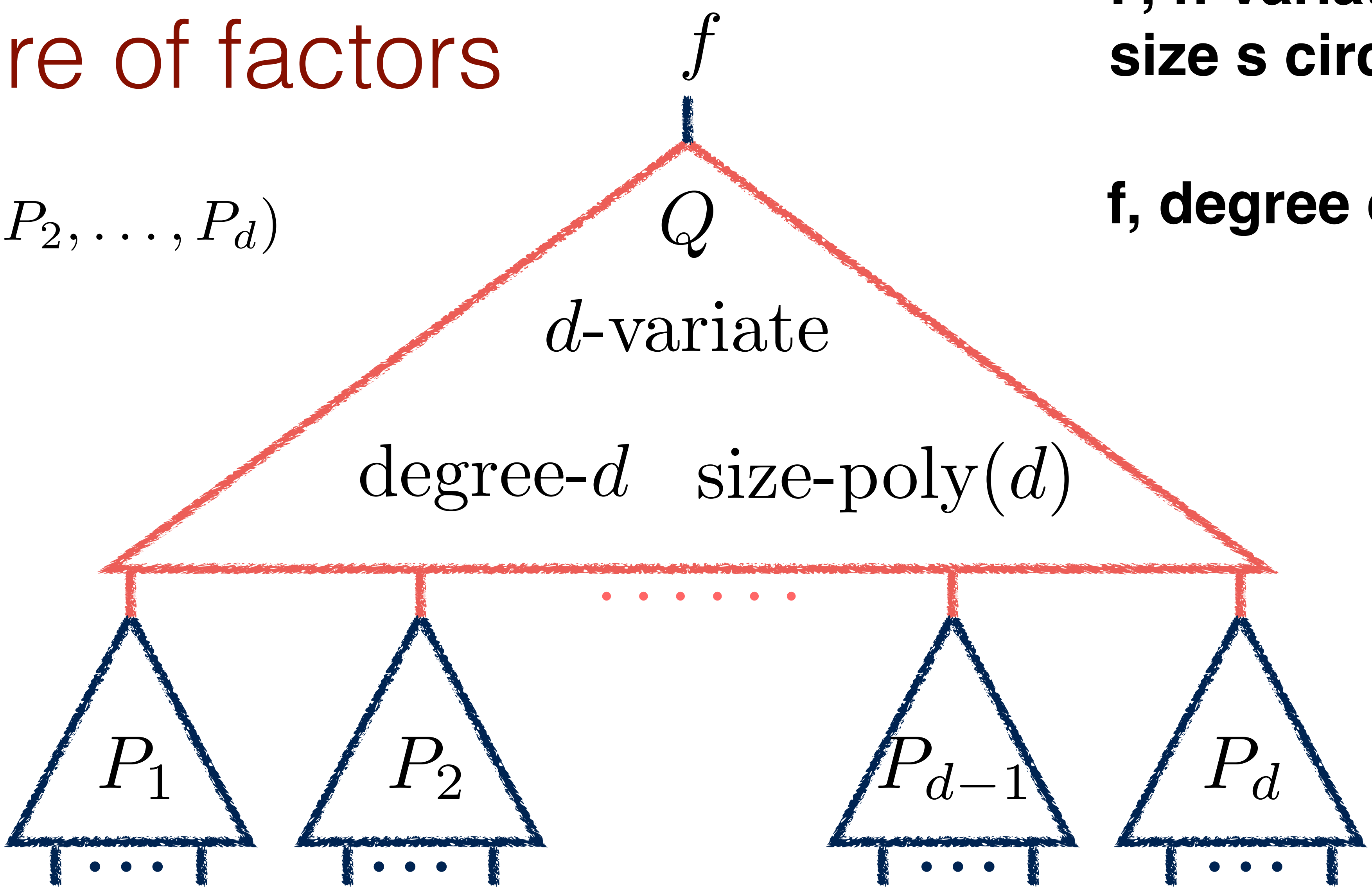

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formula $\rightarrow$ formula

VNP $\rightarrow$ VNP

Factors of poly in VNP

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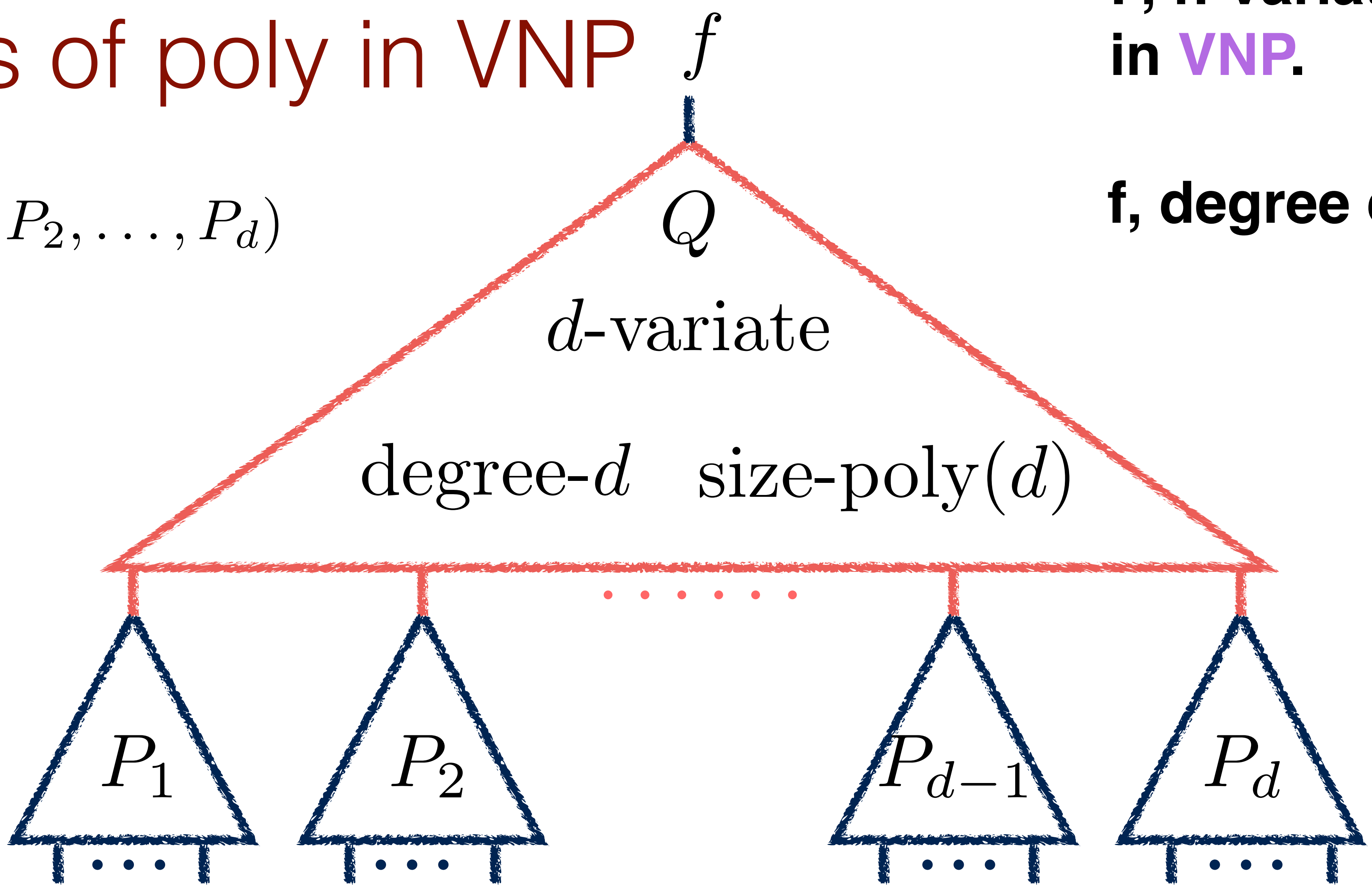

P, n-variate, degree D in VNP.

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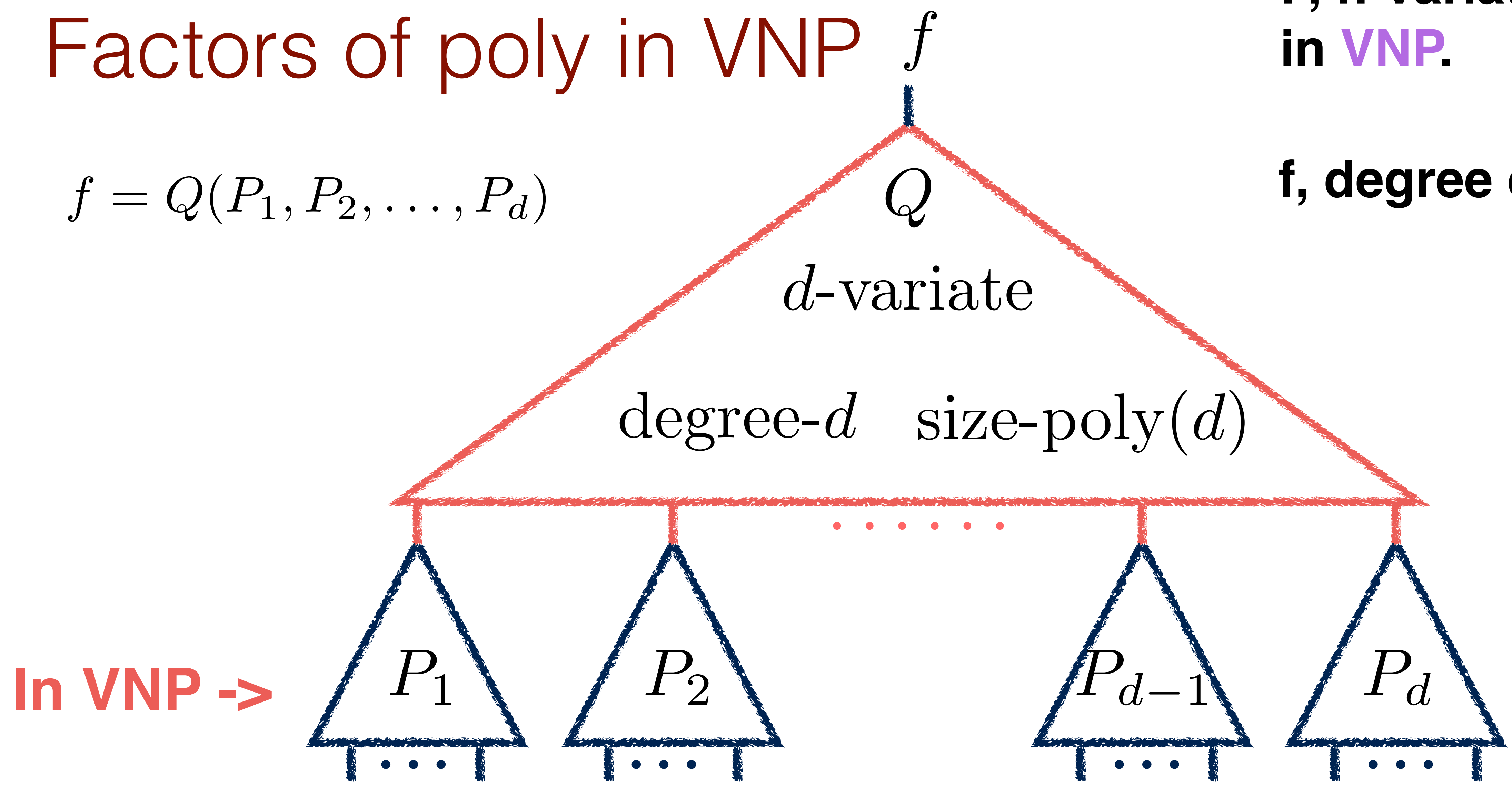

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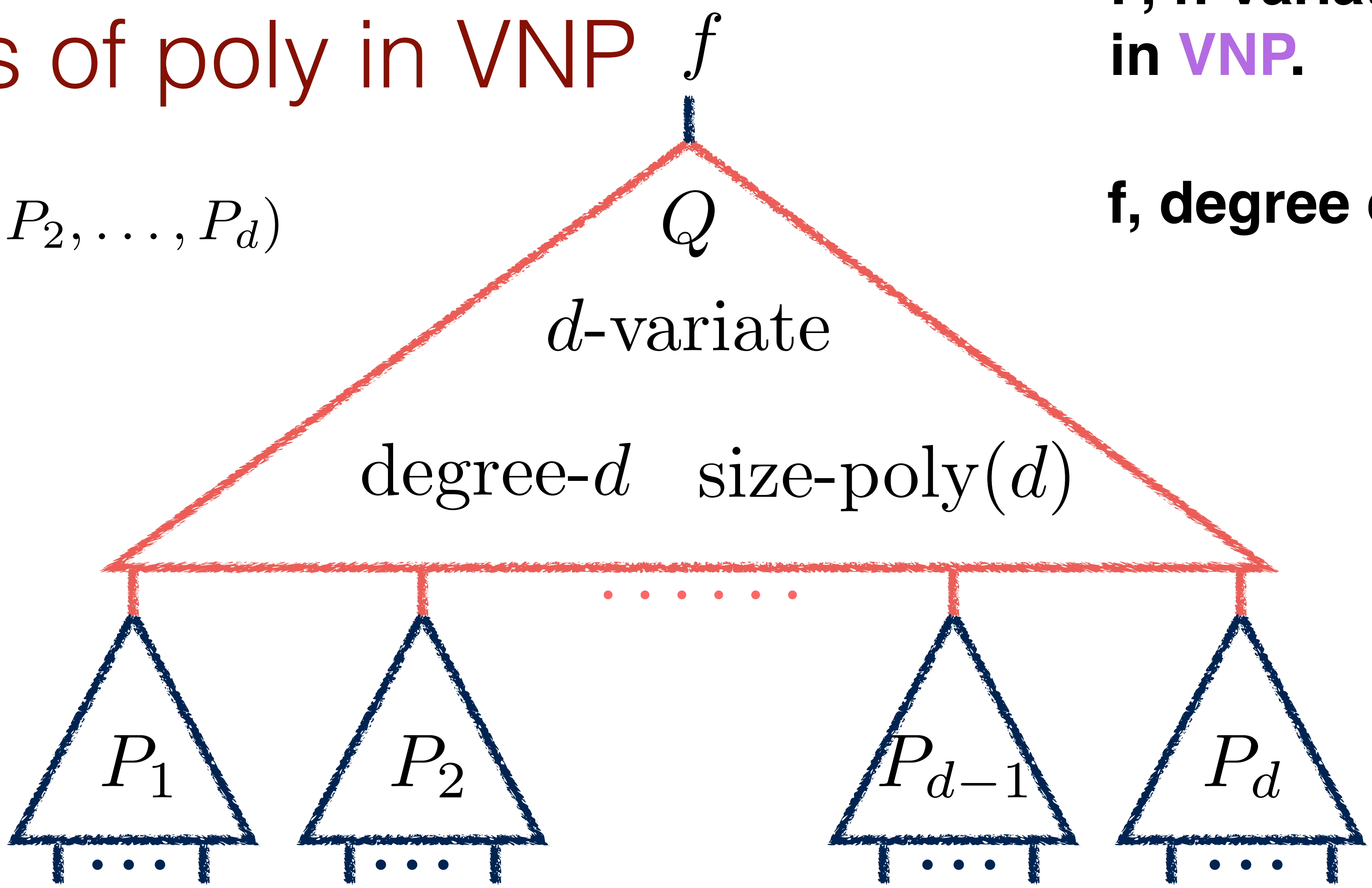

Theorem [Valiant]

If each $\mathbf{P_i}$ is in VNP, then $\mathbf{Q(P_1, P_2, \ldots, P_d)}$ is in VNP.

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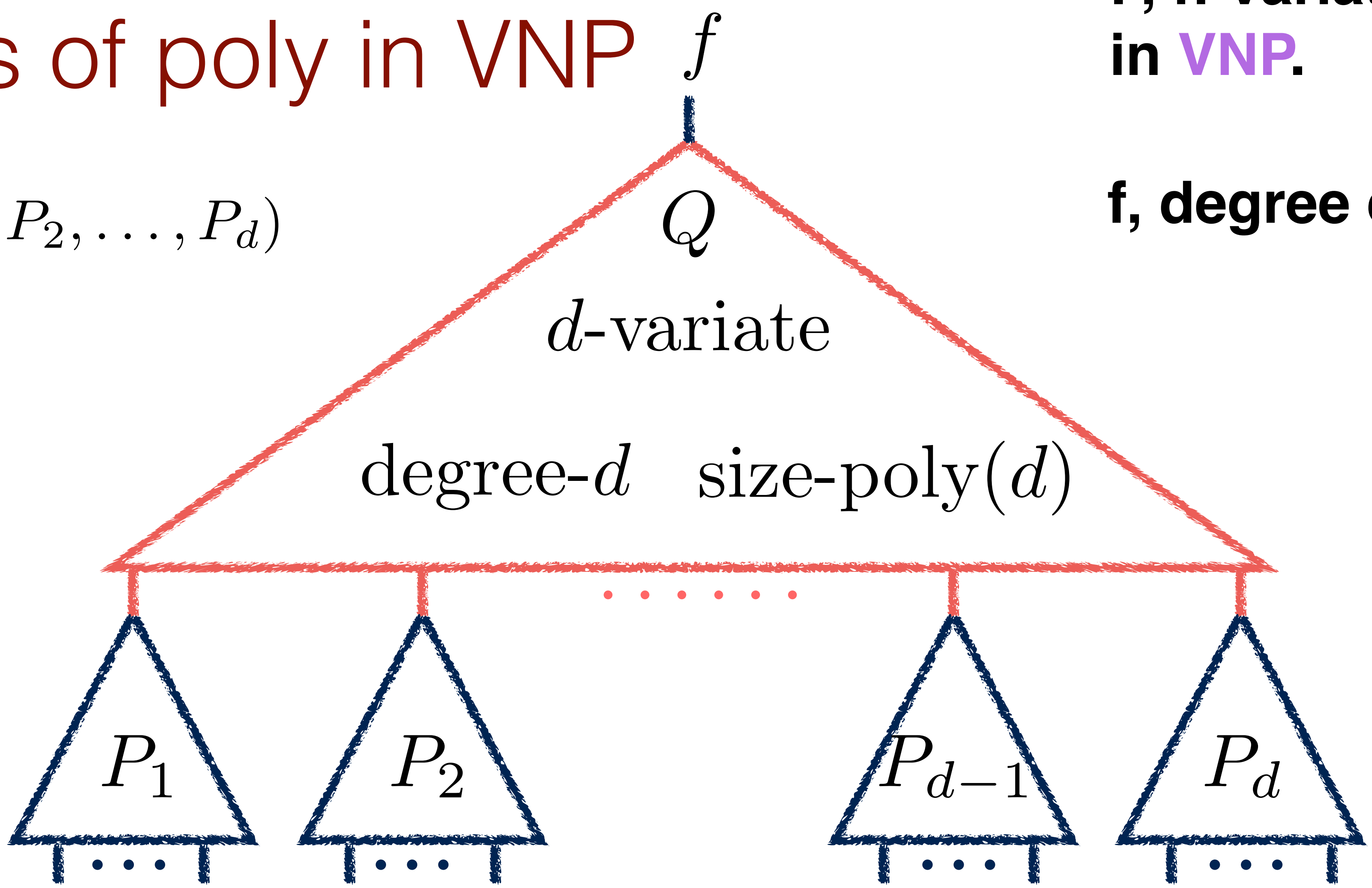

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Thus, VNP is closed under taking factors.

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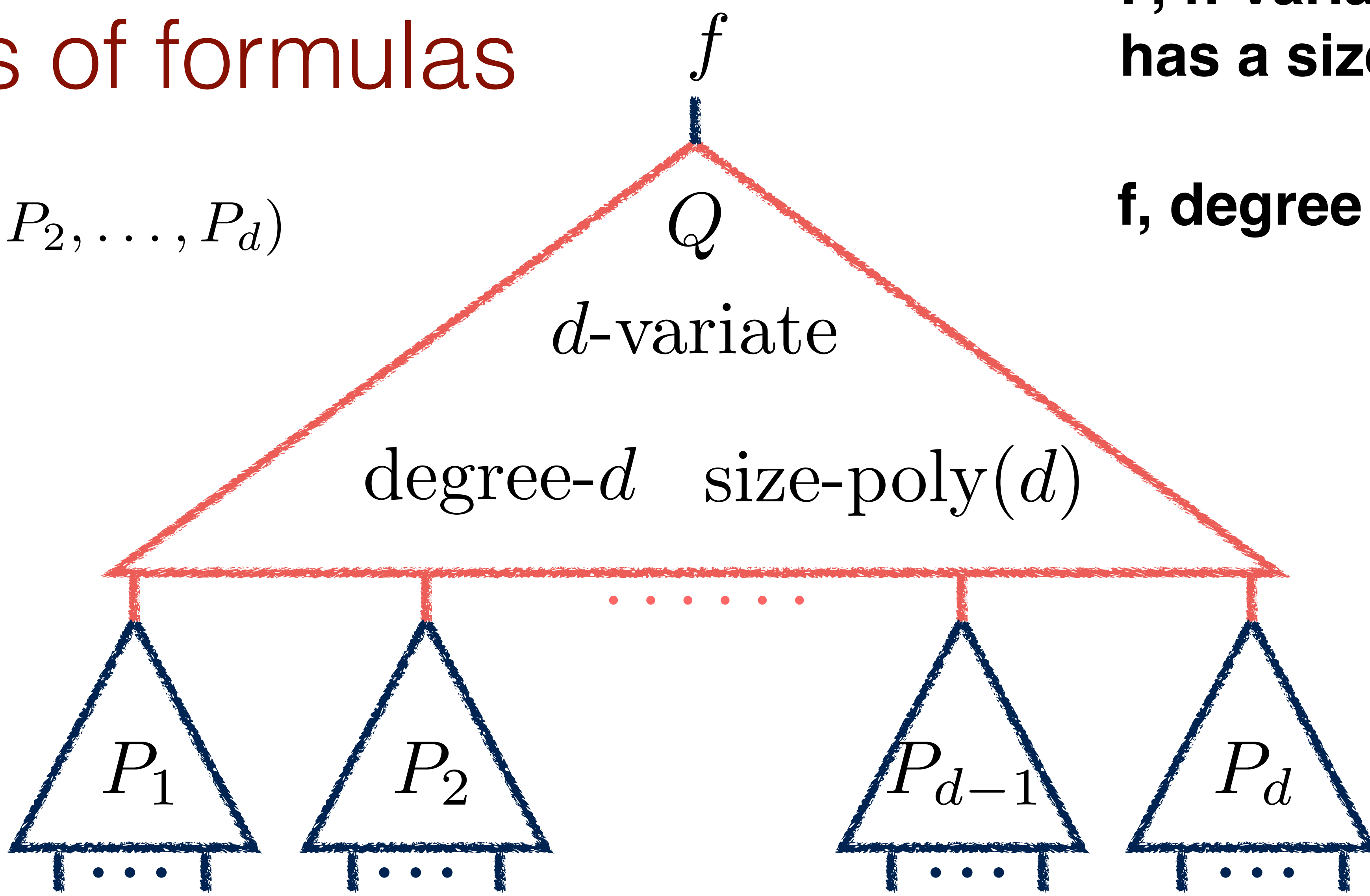

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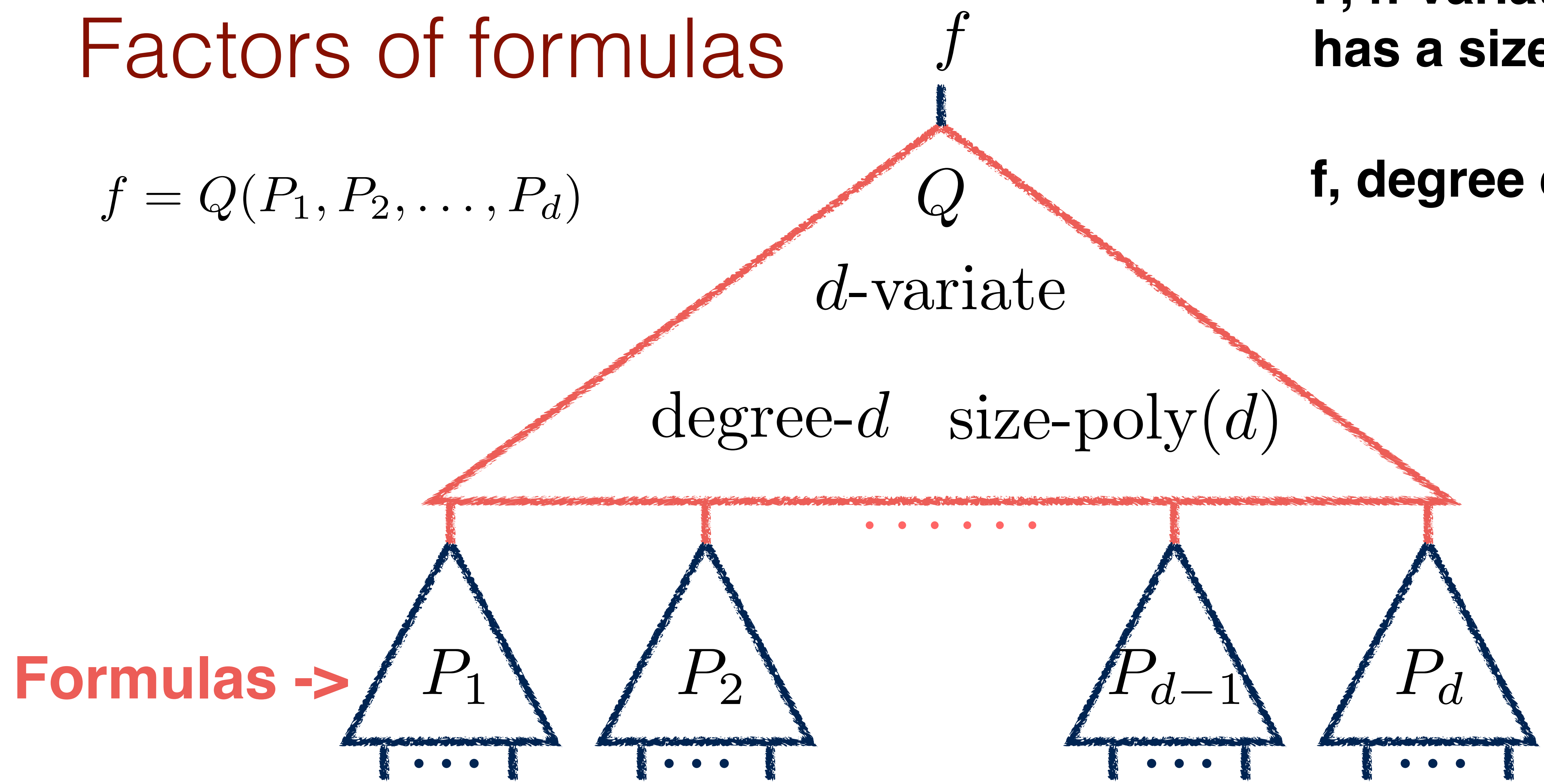

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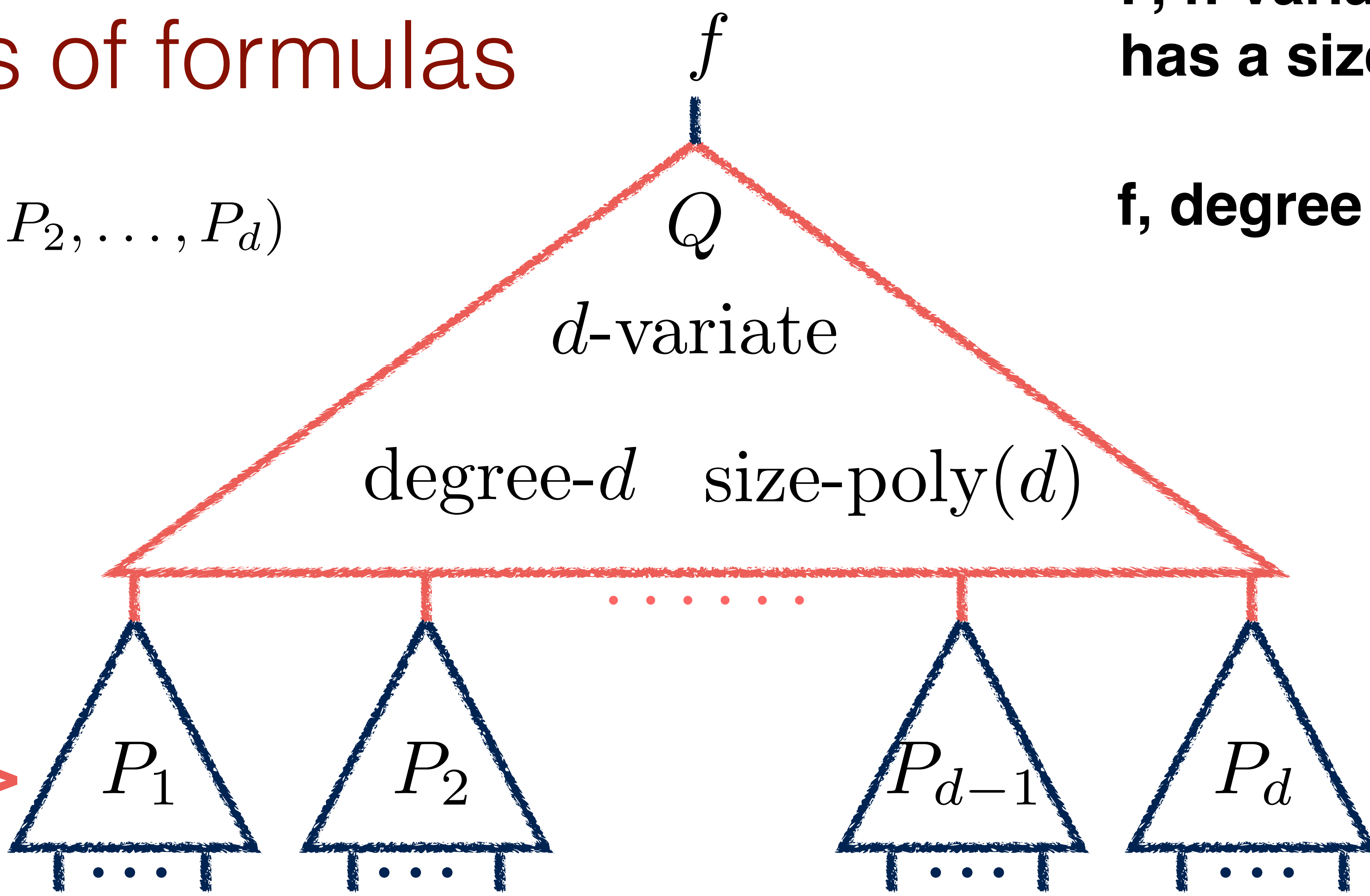

Formulas ->

Theorem [Valiant-Skyum-Berkowitz-Rackoff]

Q has a formula of size $\mathbf{d^{O(\log d)}}$.

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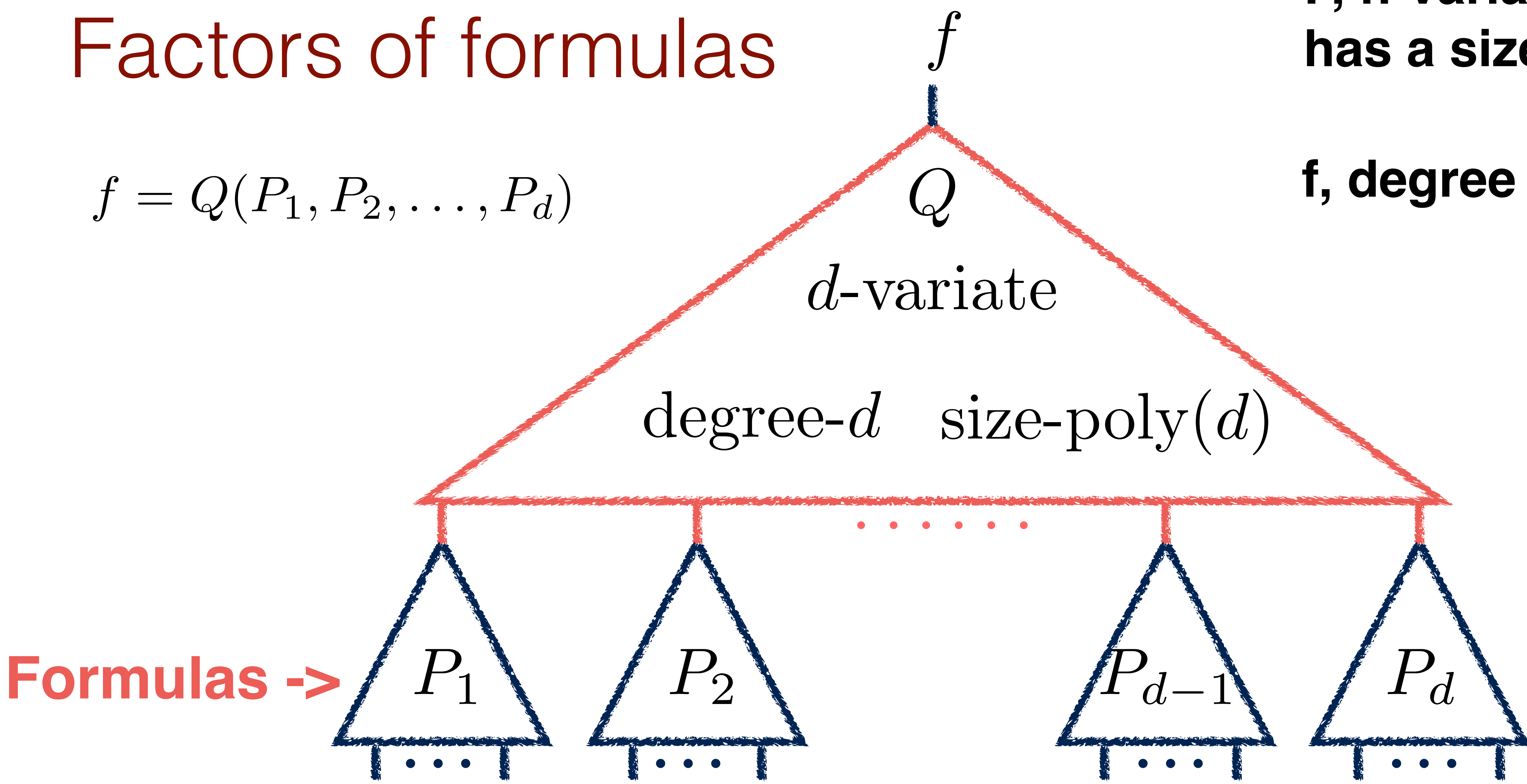

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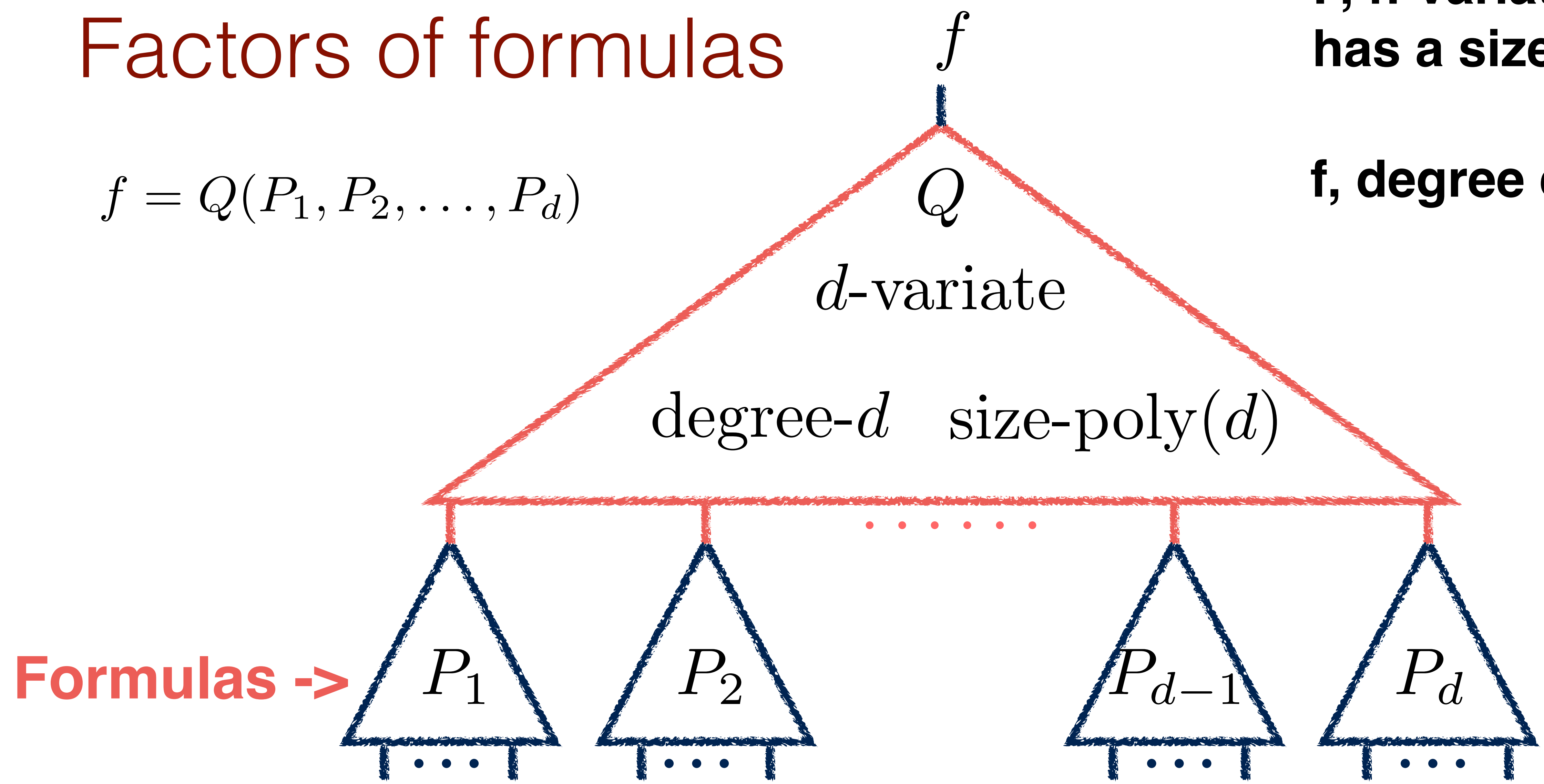

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We get a formula for f of size $\mathbf{d^{O(\log d)}} \cdot \mathrm{poly}(\mathbf{n}, \mathbf{s}, \mathbf{D})$.

Factors of shallow ckts

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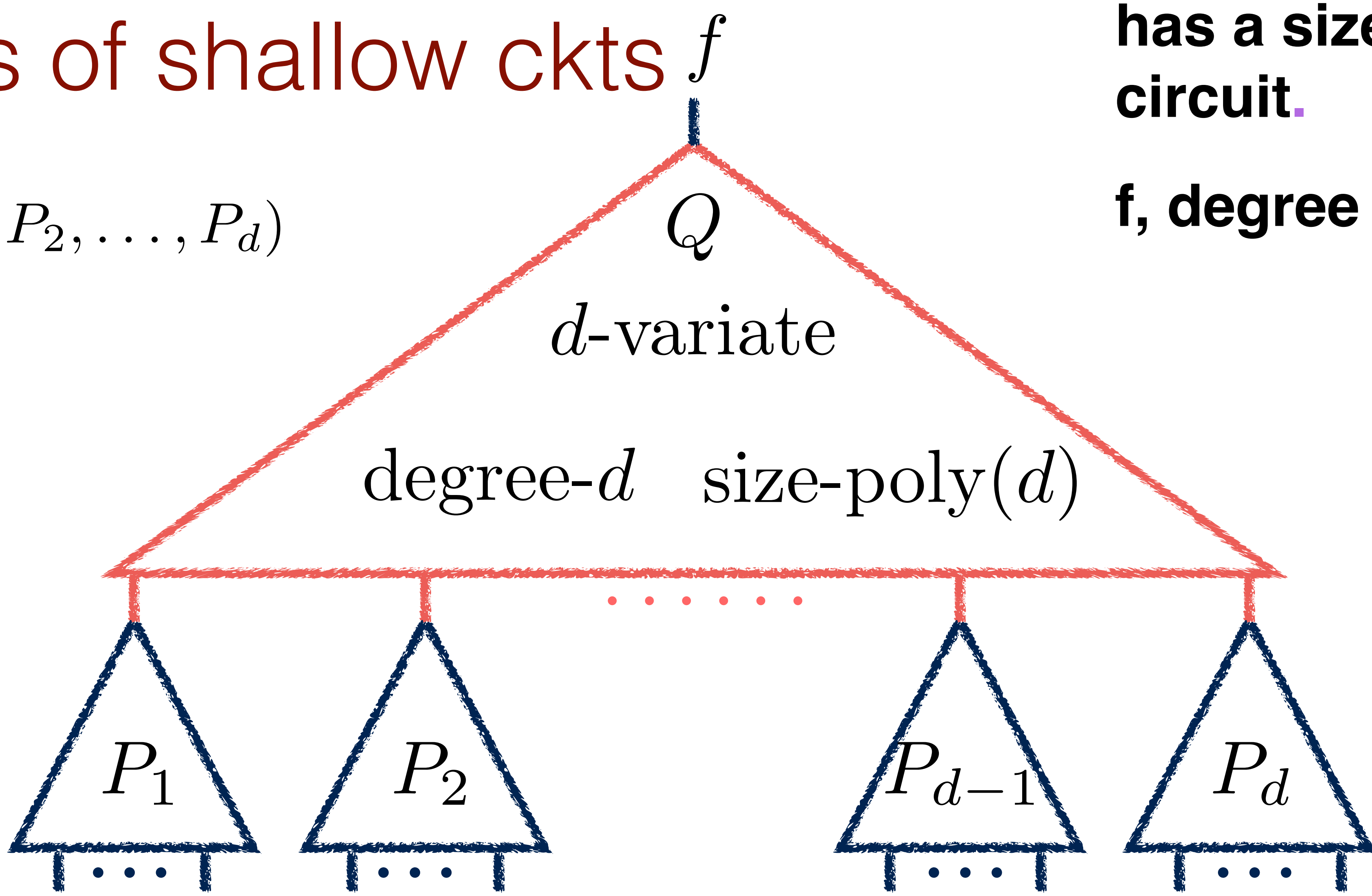

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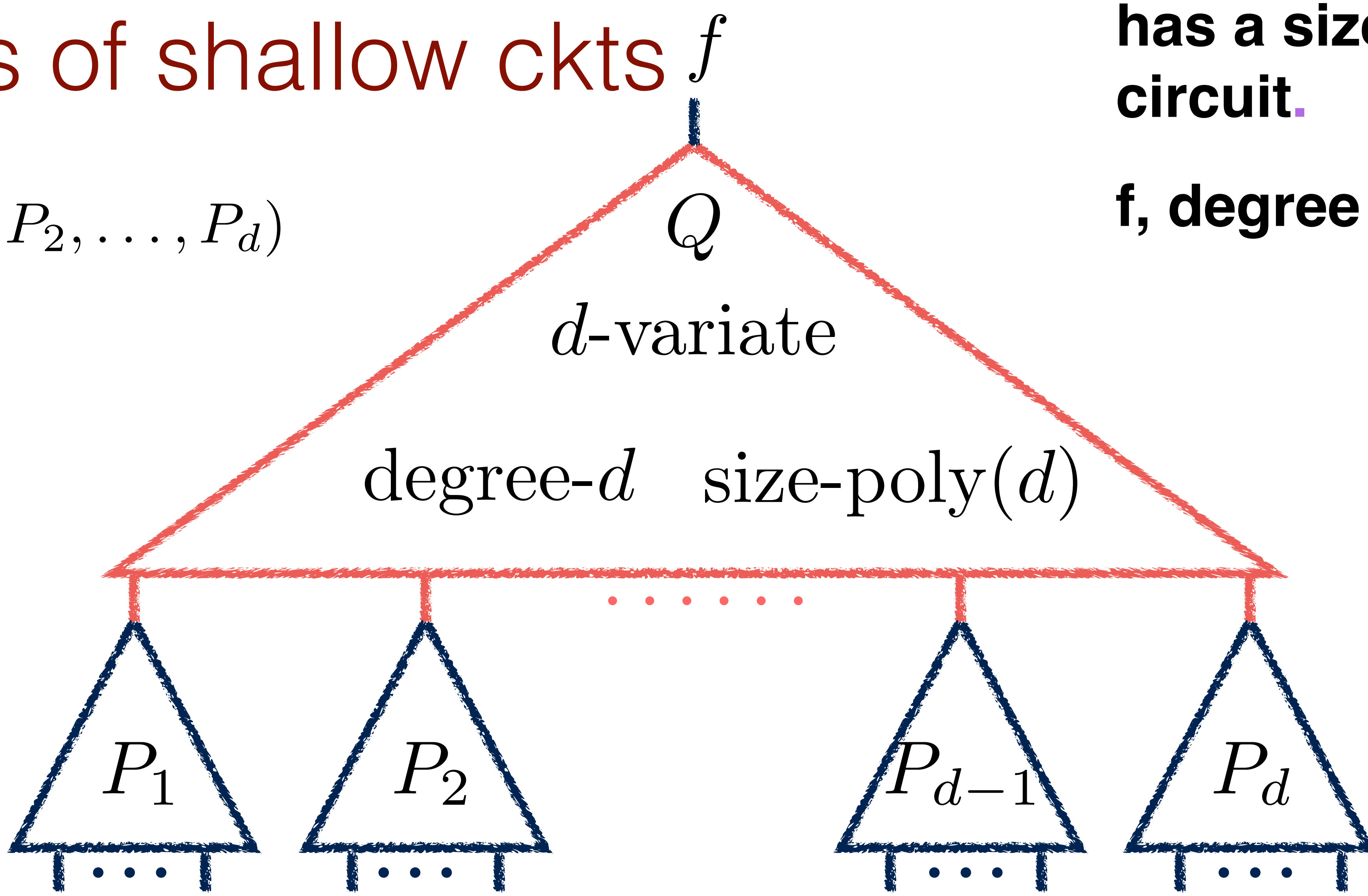

Depth k + 1 ckt ->

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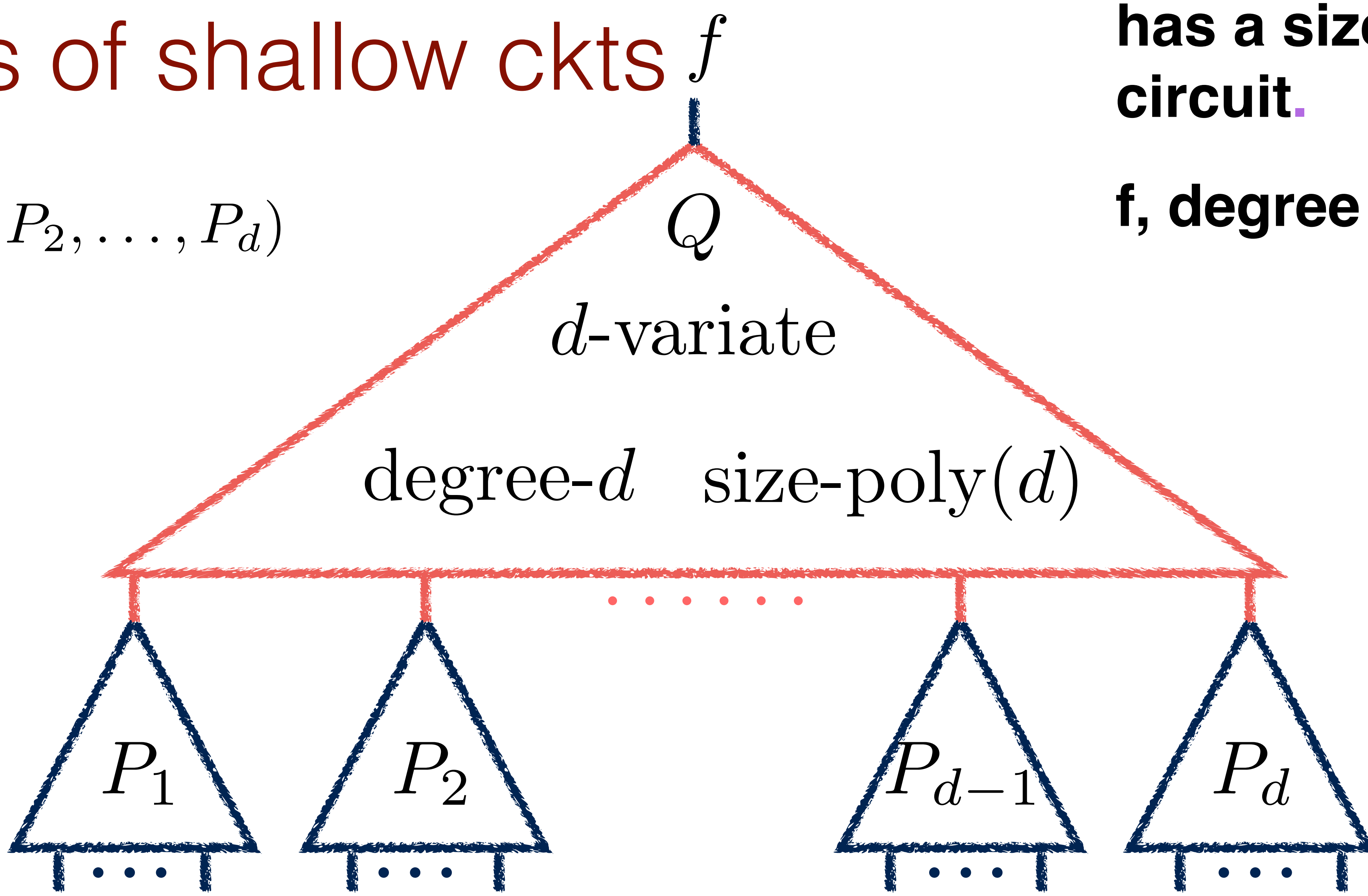

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Theorem [Agrawal-Vinay, Tavenas]

Q has a depth 2c circuit of size $\mathbf{d^{O(d^{1/c})}}$.

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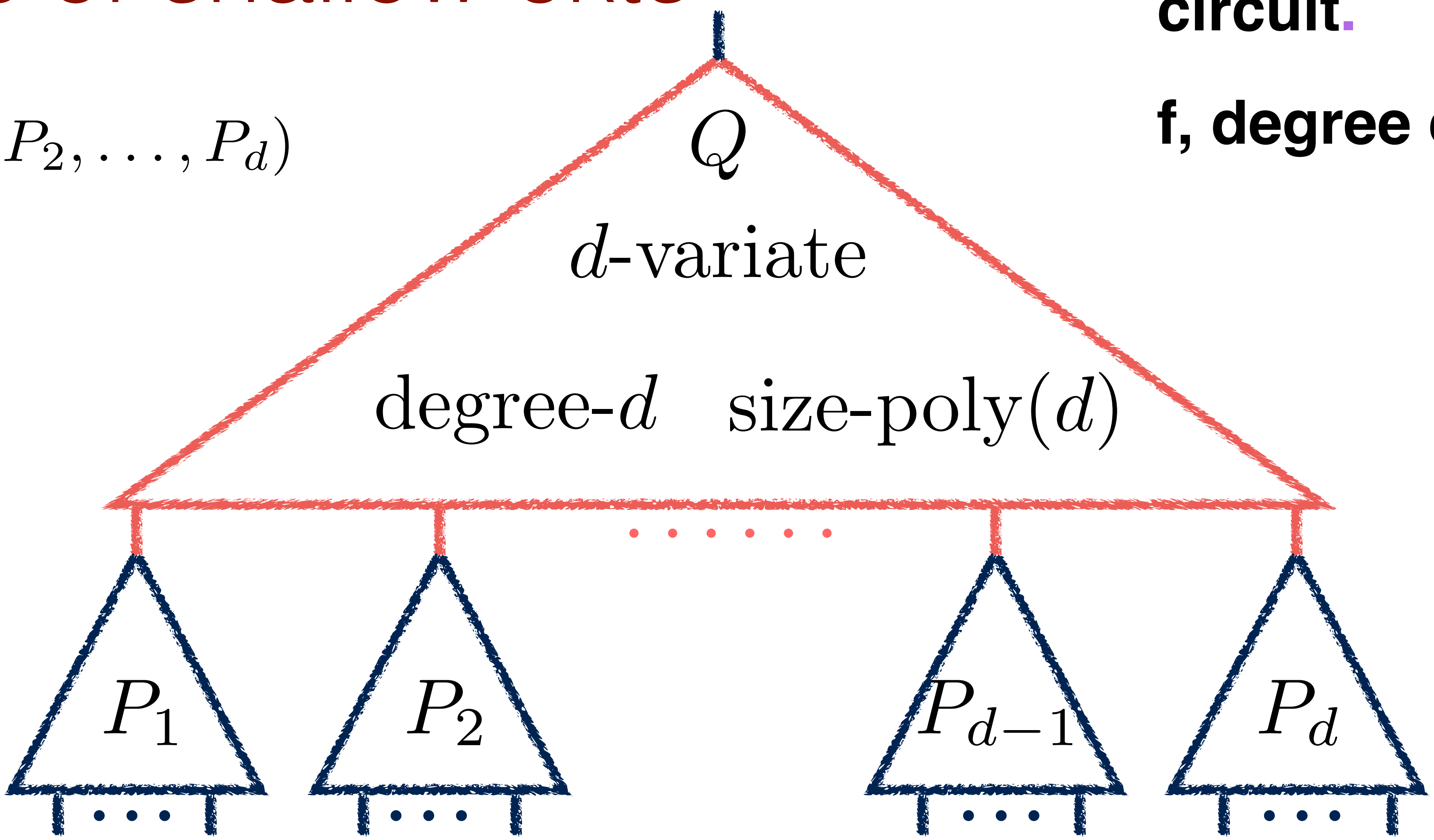

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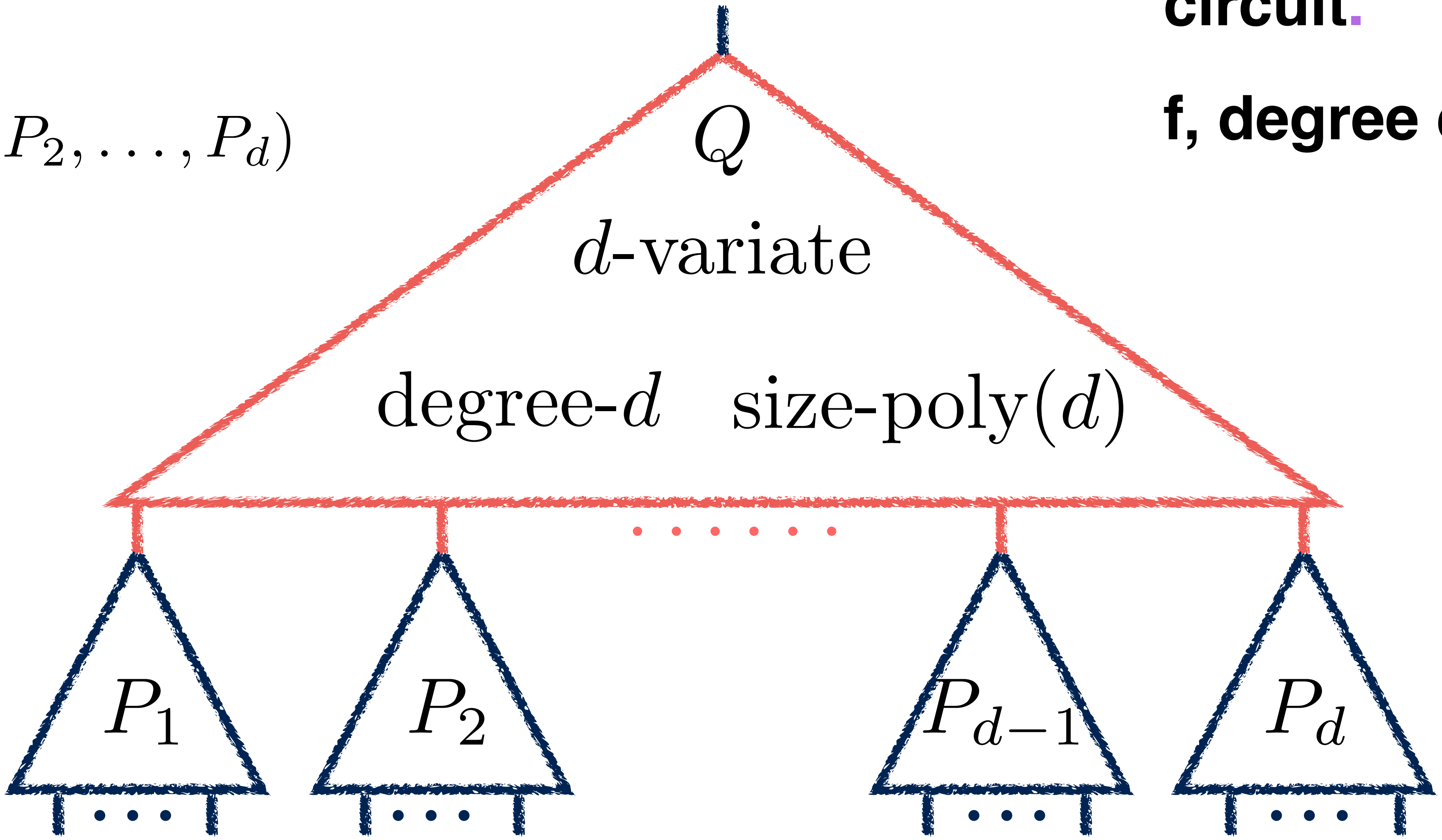

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We get a circuit for f of depth $k + 2c + O(1)$ and size $\mathbf{d^{O(d^{\epsilon})}} \cdot \mathrm{poly}(\mathbf{n}, \mathbf{s}, \mathbf{D})$

Proving the structural lemma

Structure of factors

P, n-variate, degree D size s circuit.

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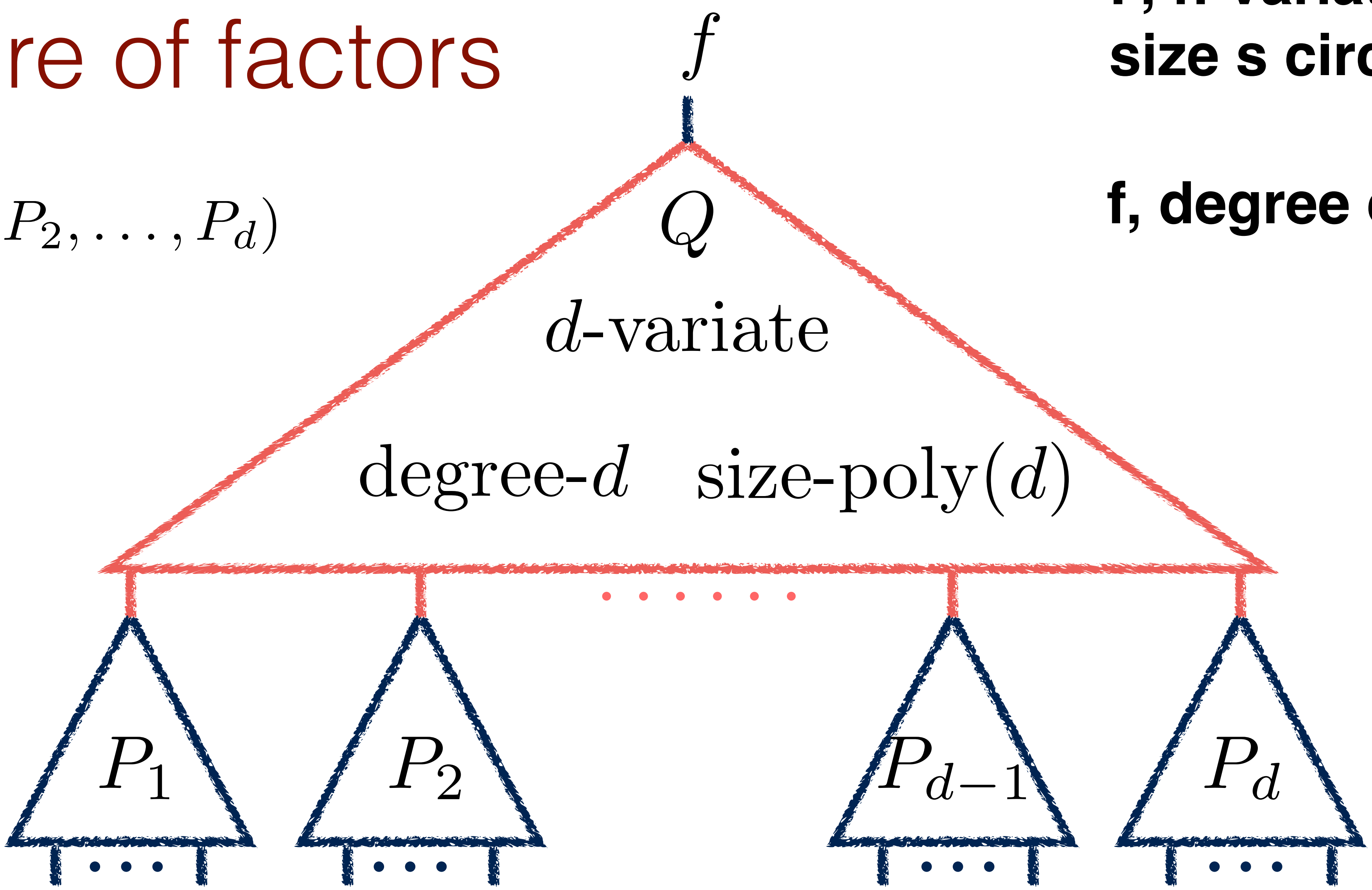

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$Q(P_1, P_2, \ldots, P_d) = f$ $\quad$ $\text{degree}(Q) = d$ $\quad$ $\text{size}(Q) = \text{poly}(d)$

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$P(X_1, X_1, \ldots, X_{n-1}, Y)$ $\quad \text{degree}(P) = r$ $\quad \text{size}(P) = s$

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Defining the generators

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$$P(\mathbf{X}, f(\mathbf{0}) + Y) = P(\mathbf{X}, f(\mathbf{0})) + Y\frac{\partial P}{\partial Y}(\mathbf{X}, f(\mathbf{0})) + \cdots + Y^r \cdot \frac{1}{r!} \cdot \frac{\partial^r P}{\partial Y^r}(\mathbf{X}, f(\mathbf{0})$$

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At the end of iteration i, we will be able to recover the homogeneous components of f of degree up to i.

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Base case : Getting a circuit for the linear term

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So, we never look beyond the first d terms. f is a function of

$P_0, P_1, \ldots, P_d$

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- These can be addressed using some standard ideas (interpolation, homogenization etc.)

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- Even somewhat non-trivial lower bounds for formulas, low depth circuits imply sub exponential time deterministic Identity Testing algorithms for them.

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Thank You!