Lower bounds on the sum of 25^{th}-powers of univariates lead to complete derandomization of PIT

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SIGTACS Webinar @CSE, IITK

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Introduction

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$$\boxed{U_{\mathbb{F}}(f, r, s) \;\geq\; \Omega(|\text{supp}(f)|^{1/r})}$$

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- (Large s). For $s \geq c \cdot (d+1)$ for *any* $c > r$, we show that

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Thus, for large s, we get $U_{\mathbb{F}}(f_d, r, s) = \Theta(d^{1/r})$, which resolves this case.

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Possible Conjecture 1

For $s \leq d$ and a constant prime-power r,

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Possible Conjecture 2

For positive constant $\delta_1 \leq 1$ and a constant prime-power r,

$$U_{\mathbb{F}}(f_d, r, d^{\delta_1}) \;\geq\; d/r$$

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Reason to choose f_d is that it is a very simple polynomial.

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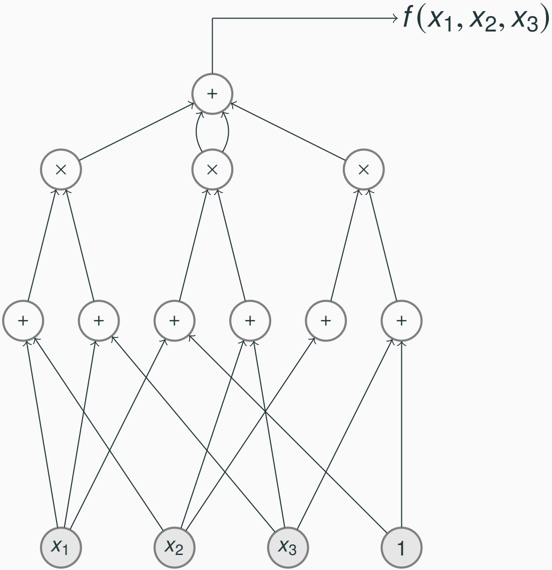
□

Conjecture C1 and Algebraic Complexity

Arithmetic Circuits



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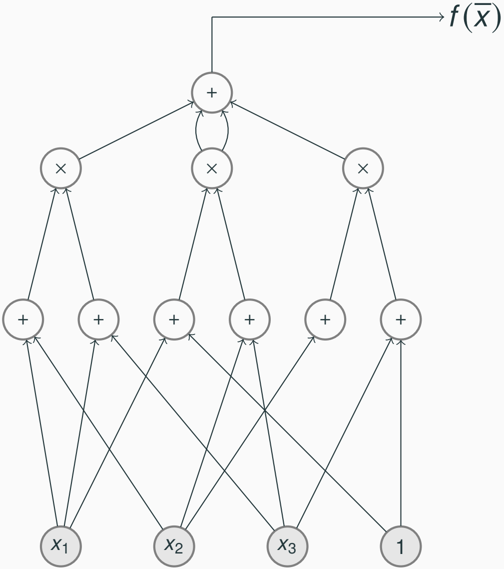

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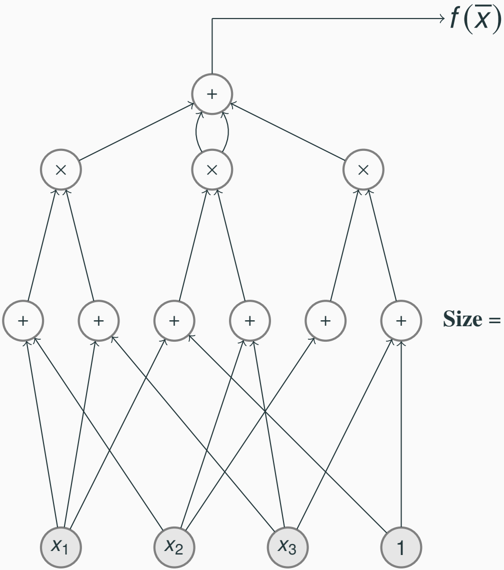

Size = number of nodes + edges

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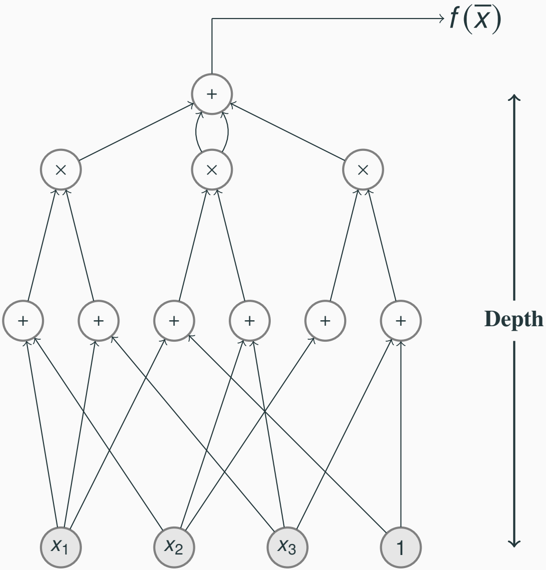

Two Important Questions

- **Valiant's Hypothesis:** Prove that symbolic perm_n requires $n^{\omega(1)}$-size circuit.

VP vs. VNP

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VP vs. VNP

- **Valiant's Hypothesis:** Prove that symbolic perm_n requires $n^{\omega(1)}$-size circuit. An *equivalent* version is: Prove $\boxed{\mathsf{VP} \neq \mathsf{VNP}}$.
- VP : A family $\{f_n\}_n \in \mathsf{VP}$ (over $\mathbb{F}$) if f_n is a $\mathrm{poly}(n)$-variate polynomial, of degree $\mathrm{poly}(n)$ over $\mathbb{F}$, computed by $\mathrm{poly}(n)$-size circuit.
- VNP : A family $\{f_n\}_n \in \mathsf{VNP}$ (over $\mathbb{F}$) if $\exists\{g_n\}_n \in \mathsf{VP}$ & $t(n) = \mathrm{poly}(n)$:

$$\boxed{f_n(\overline{x}) \;=\; \sum_{w \in \{0,1\}^{t(n)}} g_n(\overline{x}, w)}\;.$$

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This above lemma puts PIT $\in$ RP.

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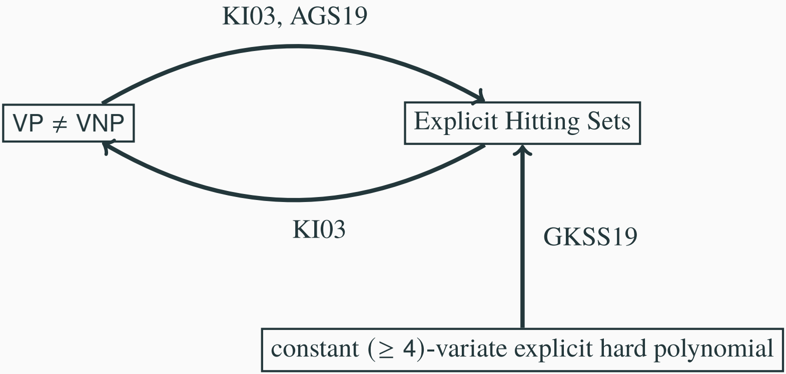

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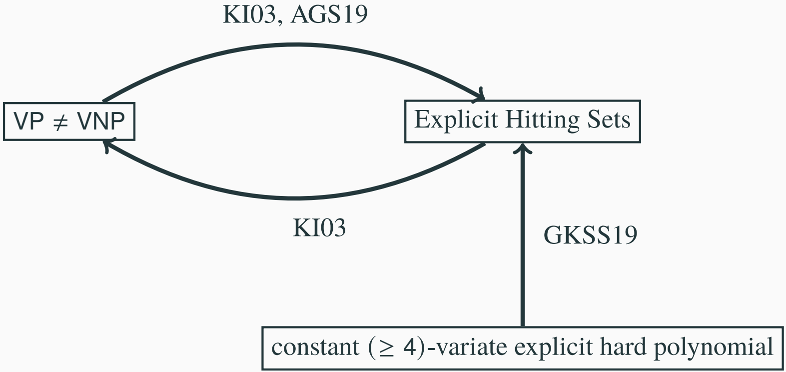

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Connecting Conjecture C1 to Algebraic Complexity

Conjecture C1 holds for an $r \geq 25$

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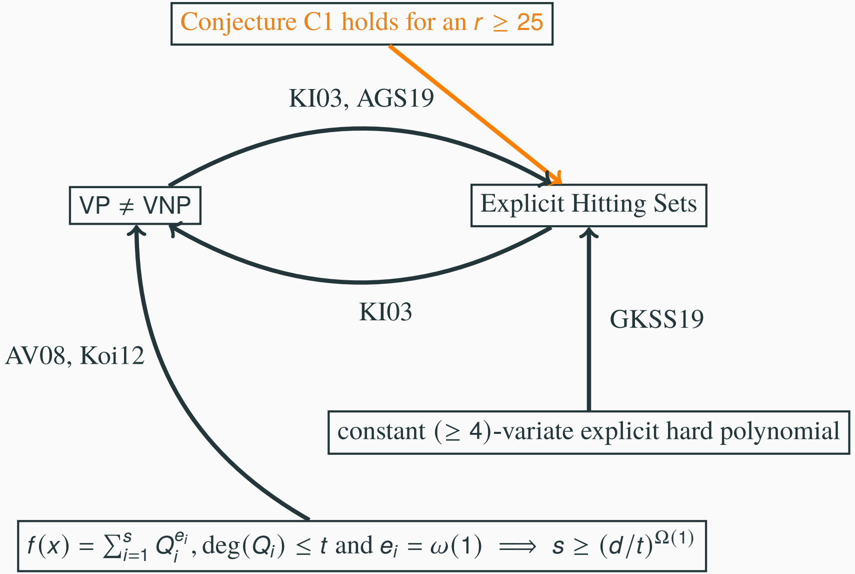

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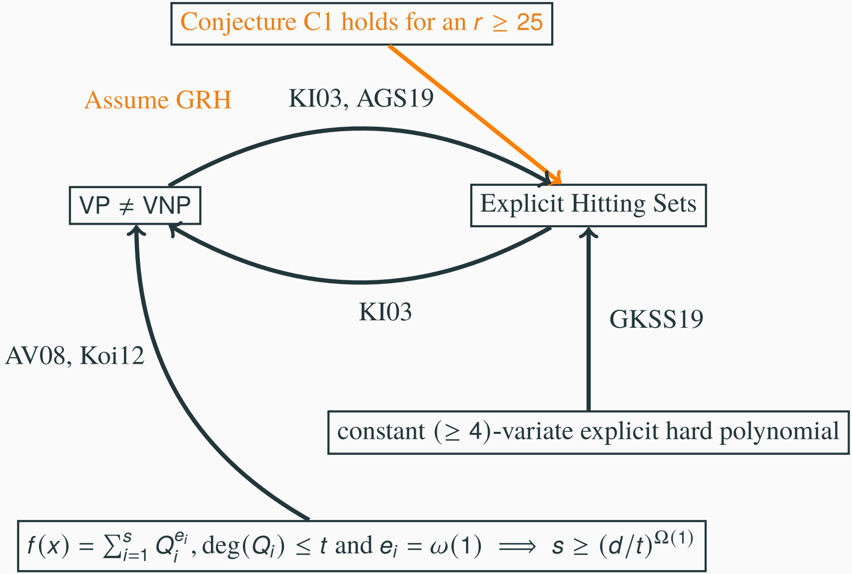

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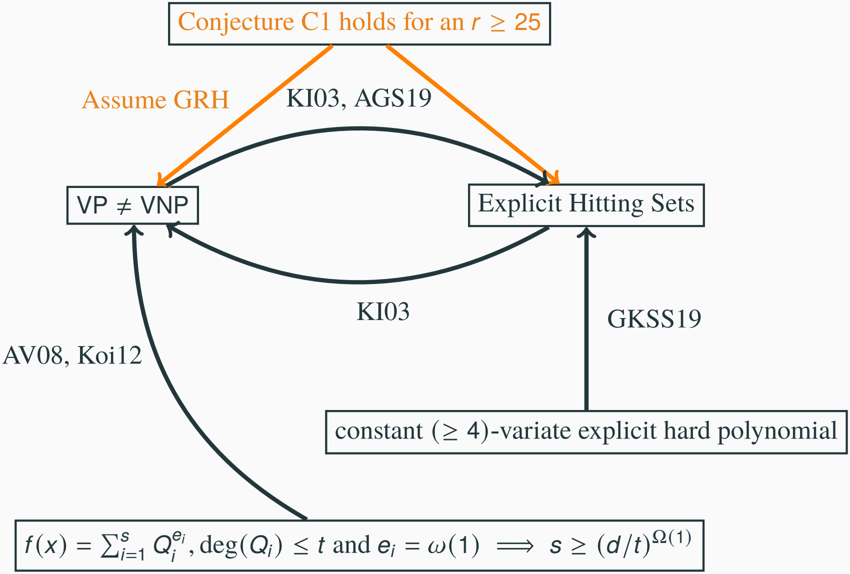

Conjecture C1 and Algebraic Complexity

Recall Conjecture C1.

Conjecture C1 and Algebraic Complexity

$$\text{C1}: \quad \boxed{(x+1)^d = \sum_{i=1}^{d^{\delta_1}} \ell_i^r \implies \Big|\bigcup_i \operatorname{supp}(\ell_i)\Big| \geq d/r^{\delta_2} = \Omega(d)}.$$

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> *Strong lower bound on sum-of-squares in non-commutative settings implies Permanent is hard [HWY11].*

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  - We could similarly conjecture (C2) that $S_{\mathbb{F}}(f_d, r, \cdot)$ is large.
  - C2 and GRH implies $\mathsf{VP} \neq VNP$; it's not clear whether it implies PIT $\in \mathsf{P}$.

Circuit Normal Form (CNF) and Algebraic Complexity

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- This circuit normal-form (CNF) has played a key role in all recent depth-reduction results [AV08, Koi12, GKKS13, Tav15].

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Note that $\deg(g_{ij}) \leq d/4$.

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CNF to sum of 25^{th}-powers

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$\mathbb{F}$ be a field of characteristic 0 or $> m$. One can write $g = \prod_{i \in [m]} g_i$ as:

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m^{th} power to sum of r^{th}-power

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Sum-Identity Lemma (DST20)

Let $\mathbb{F}$ be a field of characteristic 0 or large. Let $h(\overline{x}) \in \mathbb{F}[\overline{x}]$ and $0 \leq m \leq r$. There exist $c_{m,i} \in \mathbb{F}$ and *distinct* $\lambda_i \in \mathbb{F}$, for $0 \leq i \leq r$, such that

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Proof Idea of Main Theorems

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  - Use f_d to construct a k-variate $O(n)$ degree polynomial $P_{k,n}$ ($d := d(n)$).
- Use GKSS19: constant k-variate ($k \geq 4$) explicit hard polynomial implies blackbox-PIT $\in \mathsf{P}$.

Conjecture C1 to constant k-variate hard polynomial

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- ϕ *cannot* increase the union-support or the top fan-in.

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- f_d has sum of r-th power representation with top fan-in $s_0 := \mathrm{poly}(d^{1/\mu}, kn)$ and support-union at most $s_1 := \binom{k+kn/4}{k}$.

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- $P_{k,n}$ is hard $\implies$ PIT $\in \mathsf{P}$ (using GKSS19).
- Instead of 25-CNF, we could have used 5-CNF, then $s_1 := \binom{k+kn/2}{k}$ which is $> d$. Thus, $r \geq 25$ is *required*!

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- Thus, GRH and Conjecture C1 $\implies \mathsf{VP} \neq \mathsf{VNP}$.

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- Using Valiant's Criterion, $\{P_{k,n}\}_k \in \mathsf{VNP} = \mathsf{VP}$.

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- ϕ *cannot* increase the union-support or the top fan-in.

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- f_d has sum of r-th power representation with top fan-in $s_0 := \text{poly}(d^{1/\mu}, kn)$ and support-union at most $s_1 := \binom{k+kn/4}{k}$.

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- Instead of 25-CNF, we could have used 5-CNF, then $s_1 := \binom{k+kn/2}{k}$ which is $> d$. Thus, $r \geq 25$ is *required*!

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#StaySafe 😄