Reachability and Strong-connectivity under Failures

Keerti Choudhary
(Weizmann Institute → Tel Aviv University)

Based upon..

- Surender Baswana, Keerti Choudhary, Liam Roditty: Fault tolerant subgraph for single source reachability: generic and optimal. STOC 2016 and SICOMP 2018.

- Surender Baswana, Keerti Choudhary, Liam Roditty: An Efficient Strongly Connected Components Algorithm in the Fault Tolerant Model. ICALP 2017 and and Algorithmica 2019.

Fundamental Graph Problems

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Fundamental
Graph
Problems

Fundamental Graph Problems

Reachability

Fundamental Graph Problems

Fundamental Graph Problems

Reachability

Shortest-path

Fundamental
Graph
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Max-flows

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minimum-cut

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Connectivity

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strong-connectivity

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minimum-cut

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strong-connectivity

Matching

Fundamental Graph Problems

Reachability

Shortest-path

Fundamental Graph Problems

Max-flows

minimum-cut

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strong-connectivity

We already have efficient solutions..

Matching

Fundamental Graph Problems

Reachability

Shortest-path

Max-flows

Fundamental Graph Problems

minimum-cut

Connectivity

strong-connectivity

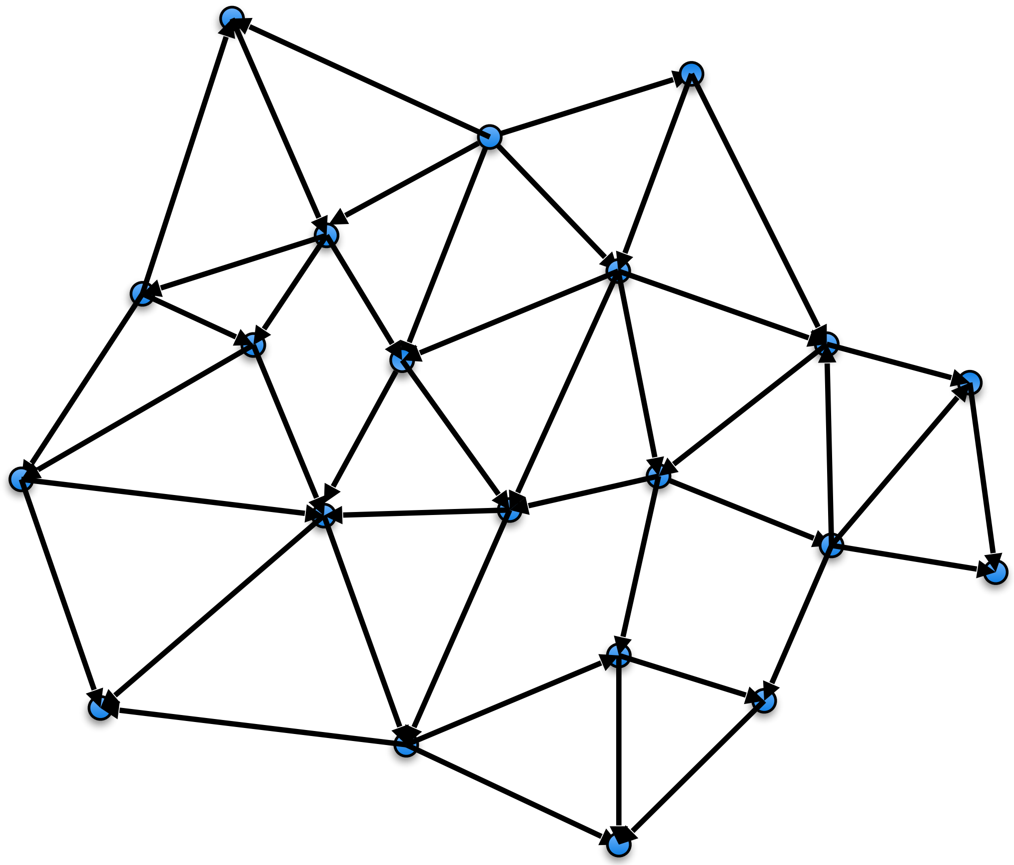
What if there are faults?

Matching

Fault Tolerant Model

Fault Tolerant Model

G



Fault Tolerant Model

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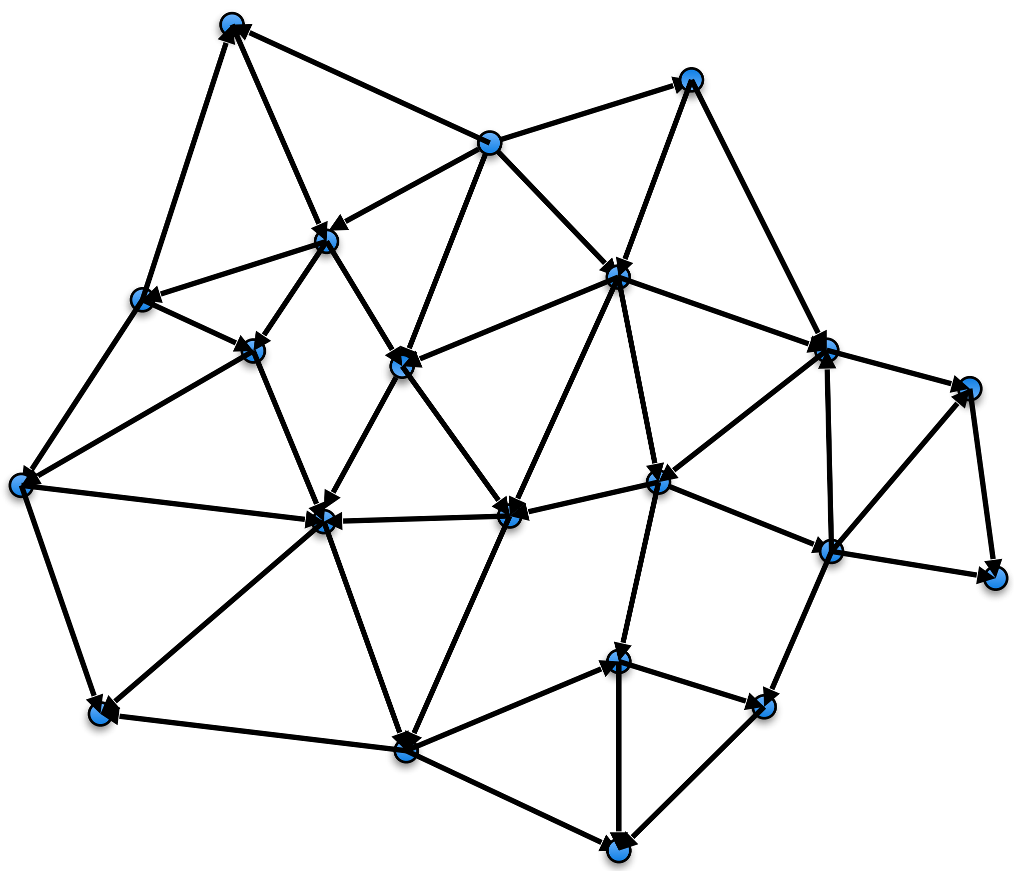
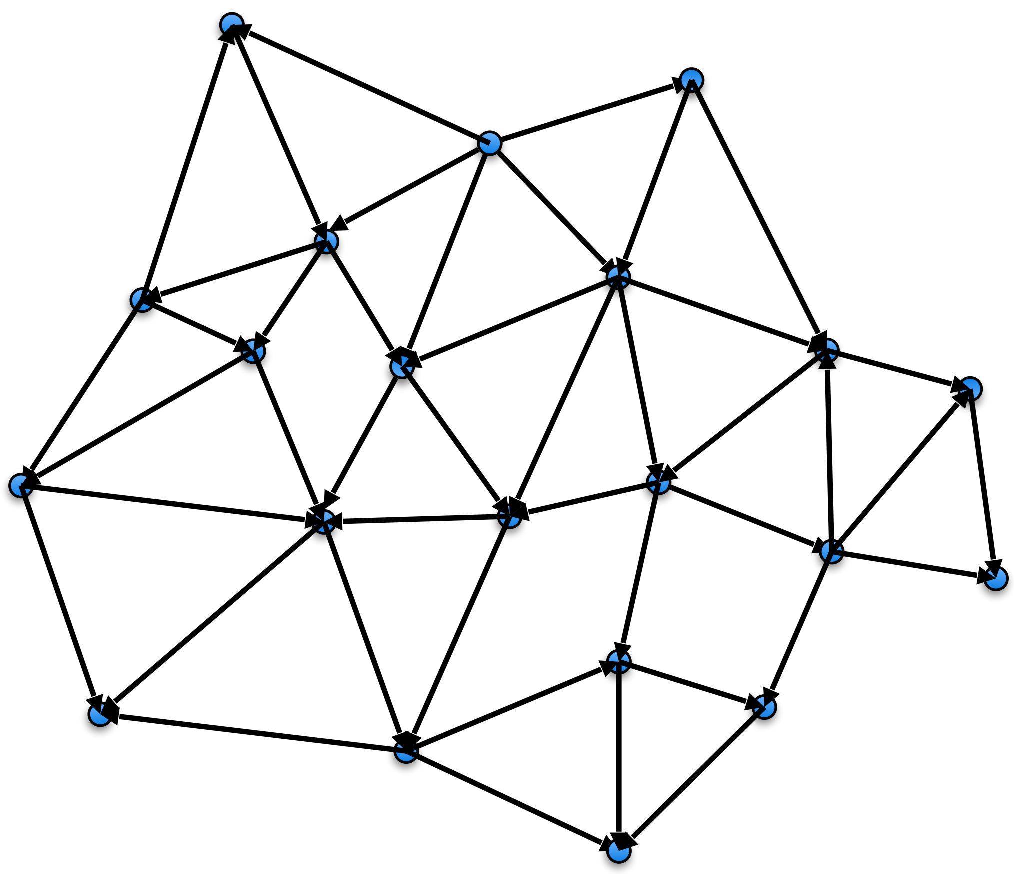


$k = 2$(Faults)

Fault Tolerant Model

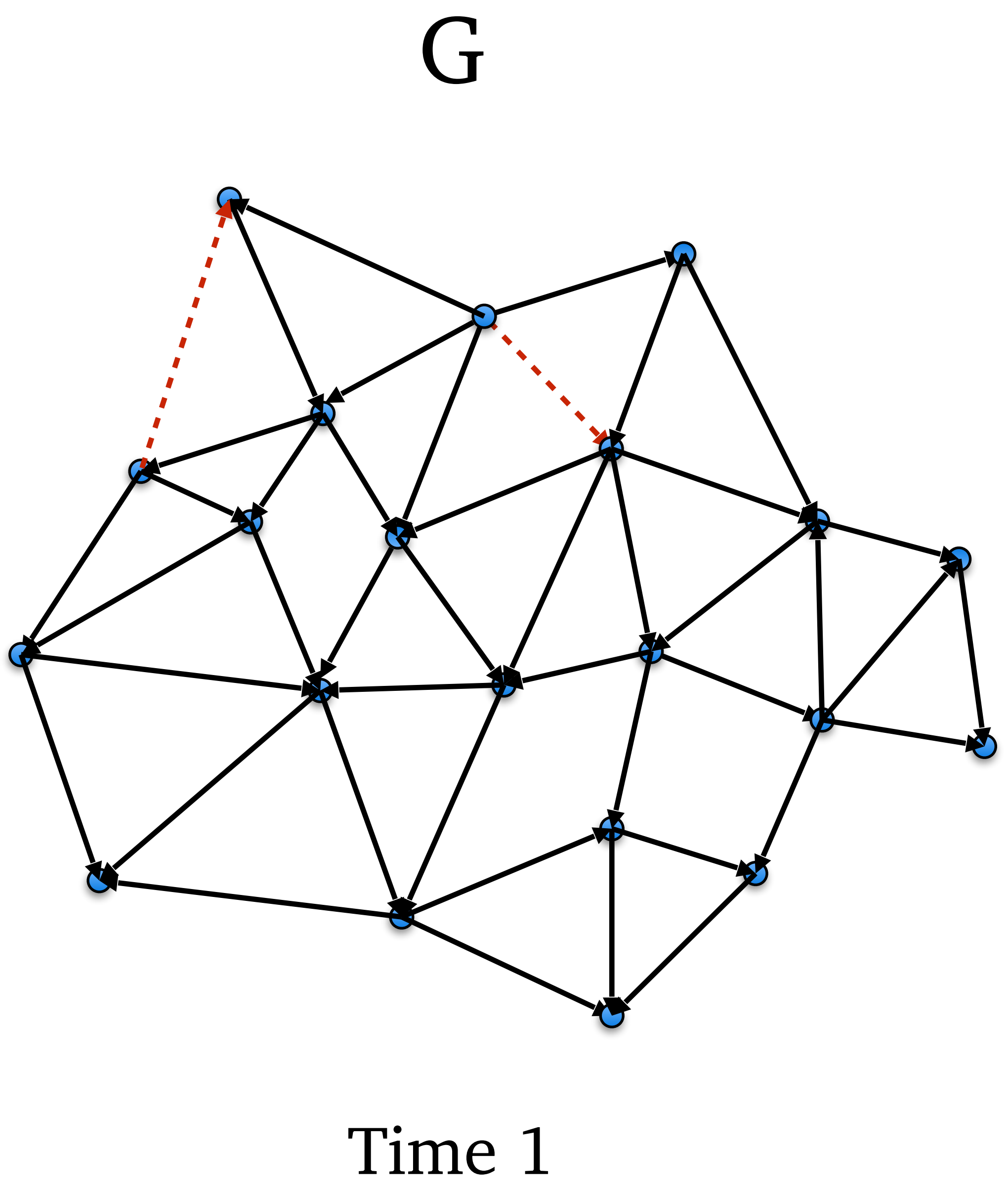
G



Time 0

$k = 2$(Faults)

Fault Tolerant Model



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Fault Tolerant Model

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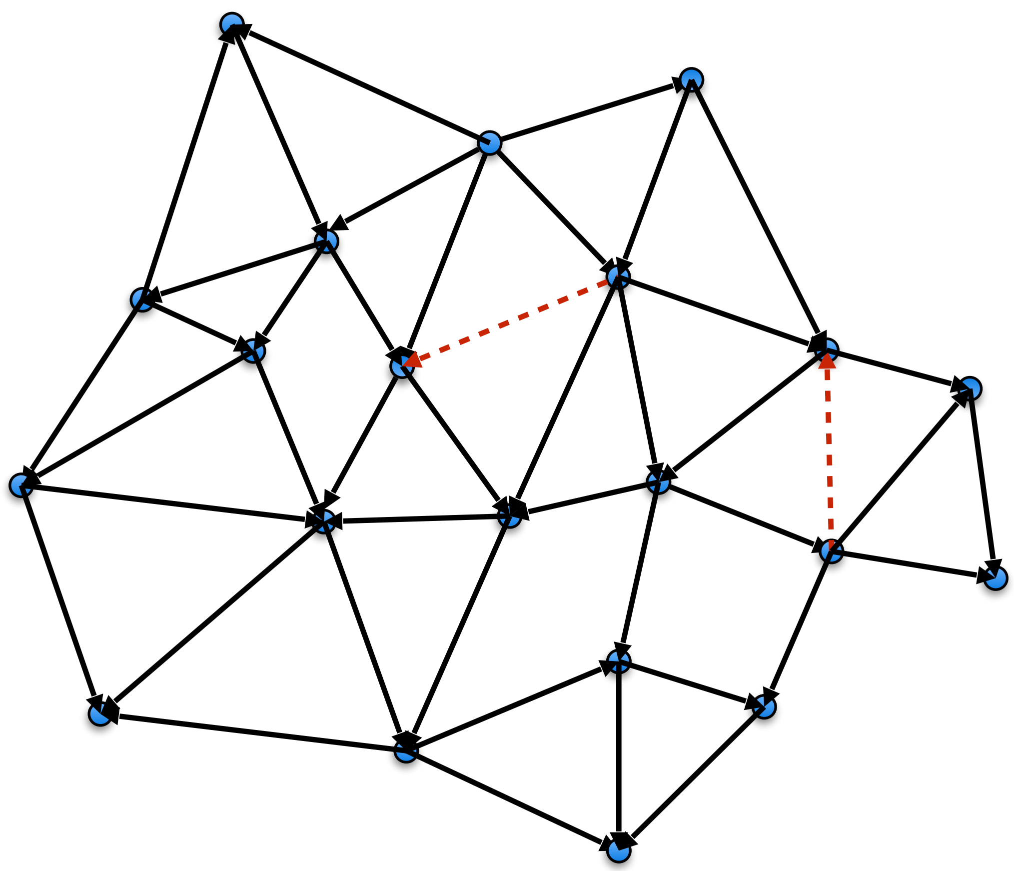

Time 2

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Fault Tolerant Model

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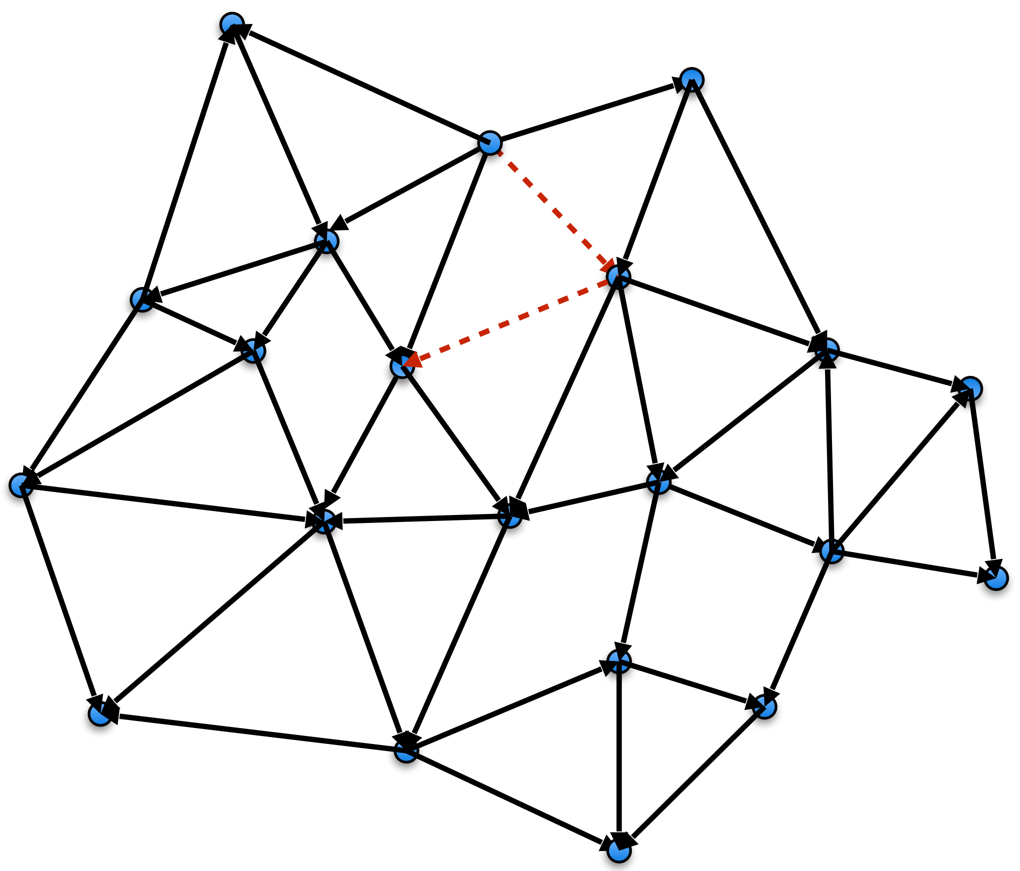

Time 3

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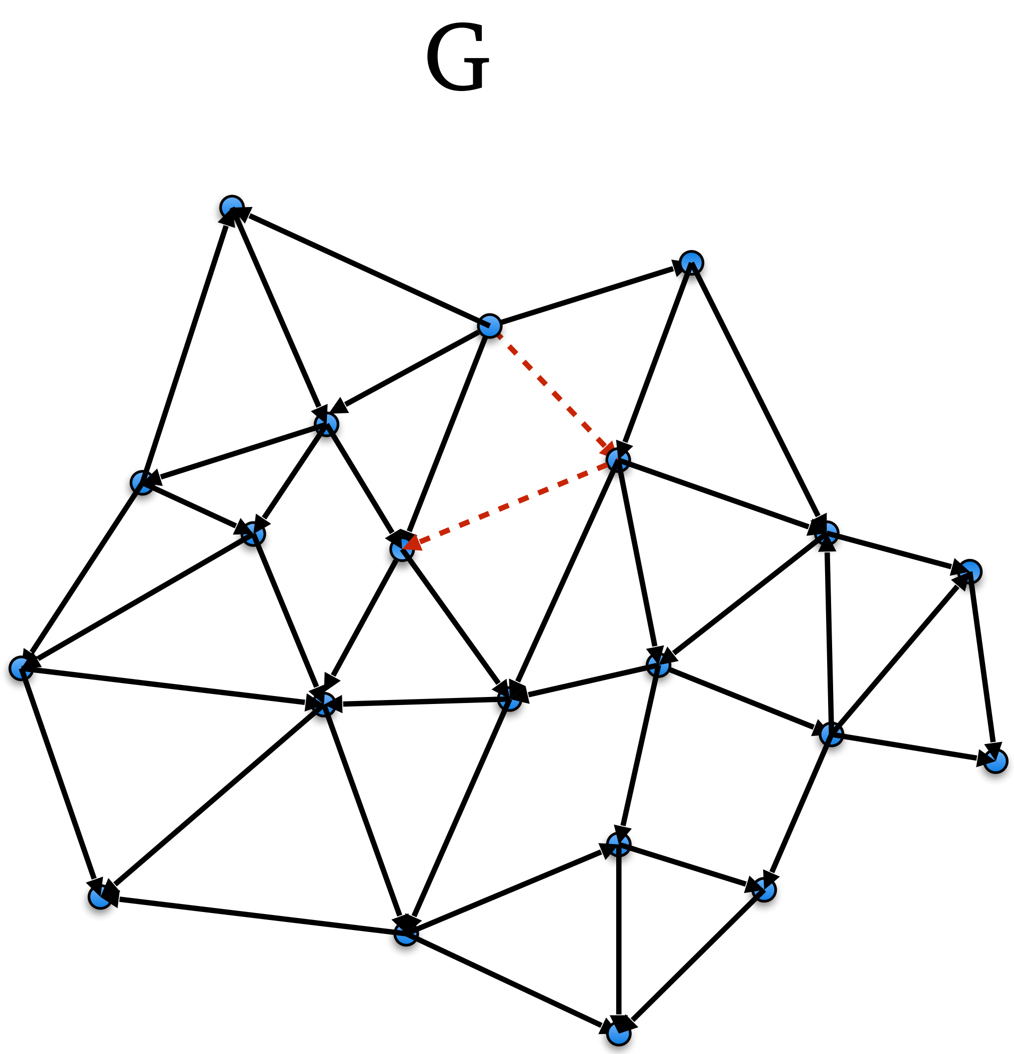
Fault Tolerant Model



Time 3

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Answer queries of the form:

- Exact/approximate distances
- Maximally Independent Set
- Minimum Spanning-tree

Fault Tolerant Model

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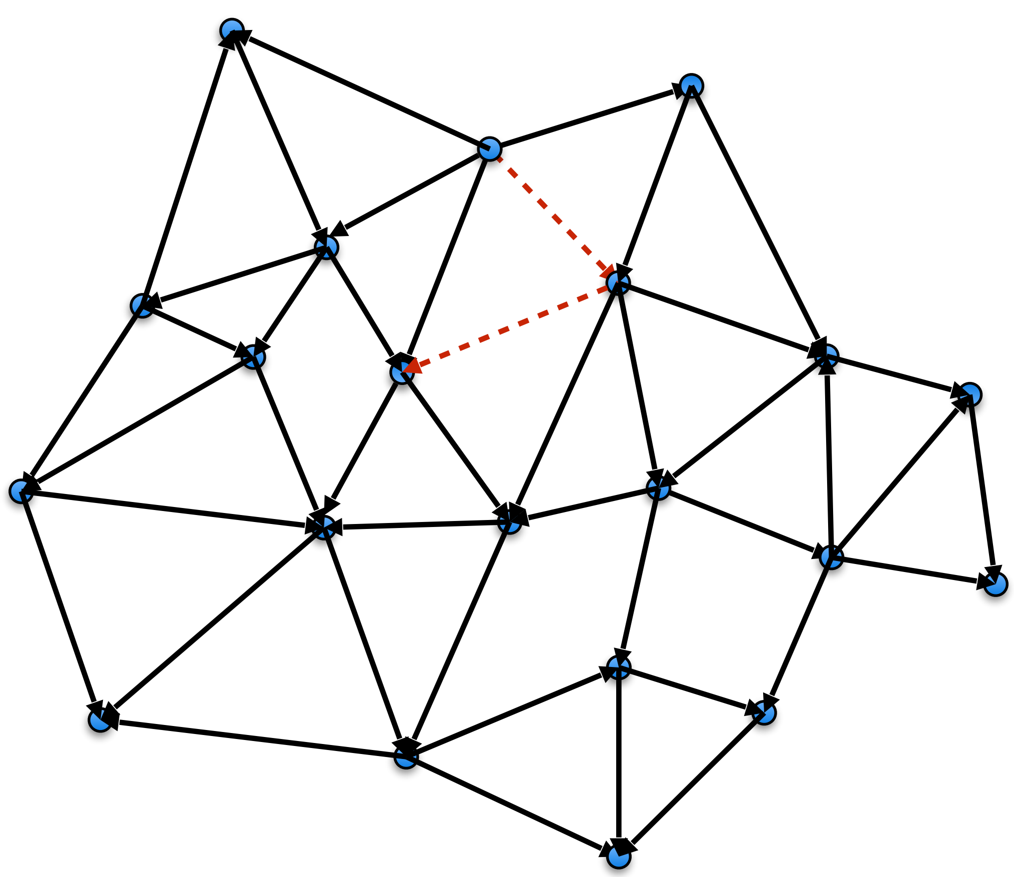

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Naive approach
Re-compute the solution
at each time.

Fault Tolerant Model

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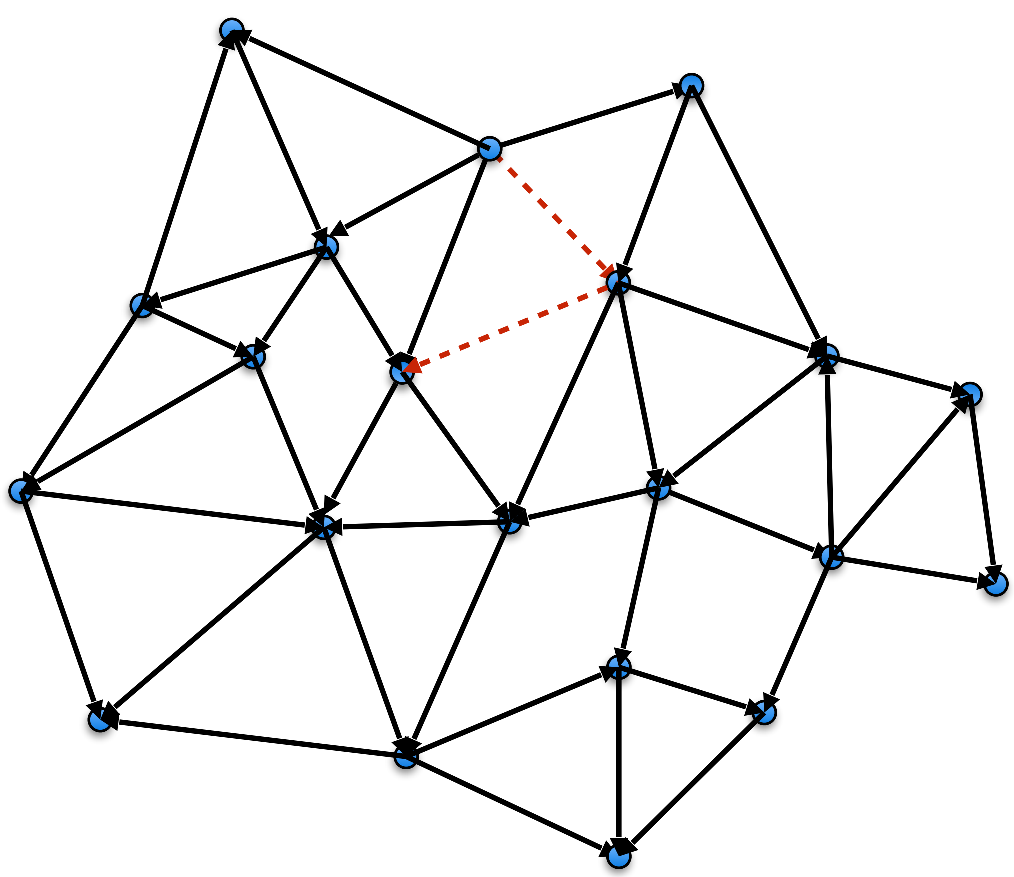

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O(m) at each step!

Fault Tolerant Model

Fault Tolerant Model

Why should we learn this model?

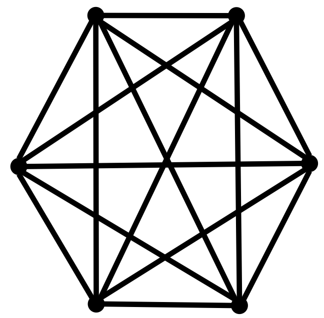
Fault Tolerant Model

Why should we learn this model?

Dynamic graph algorithms models:

- Fully dynamic – An update is an edge insertion or deletion
- Decremental – An update is an edge deletion
- Incremental – An update is an edge insertion

Dynamic Model: An Example



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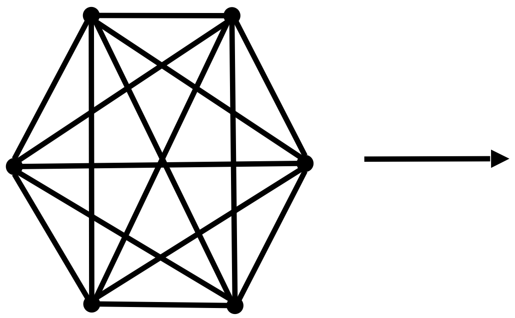

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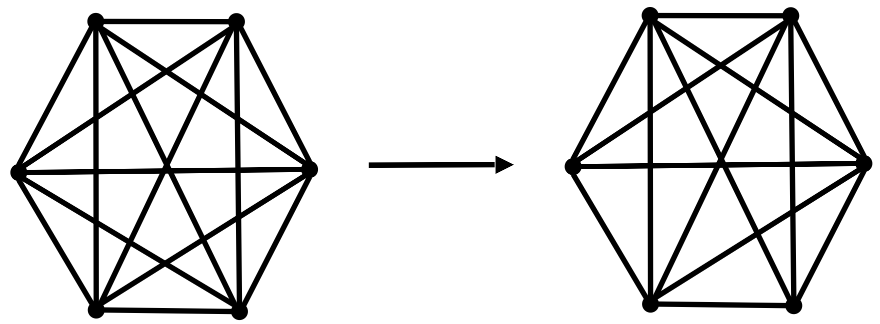

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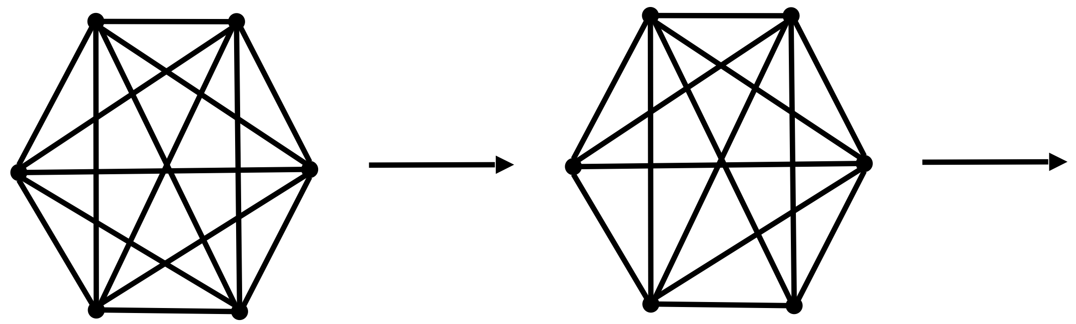

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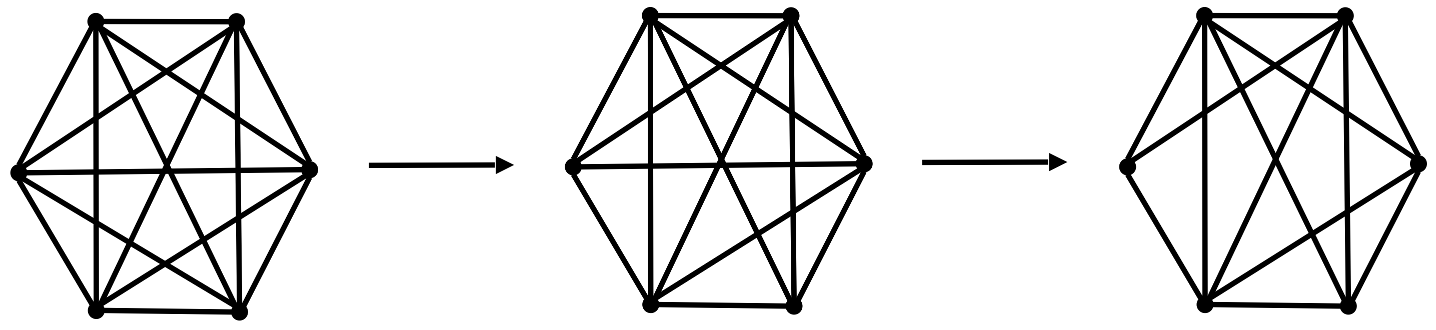

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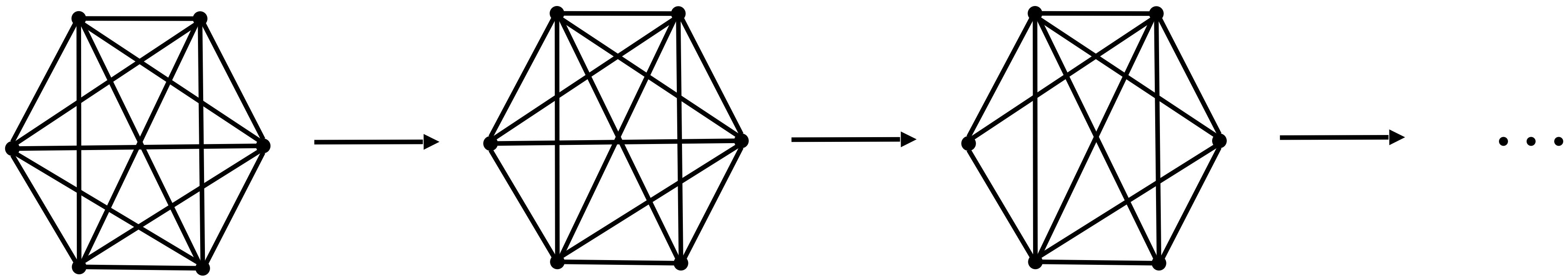

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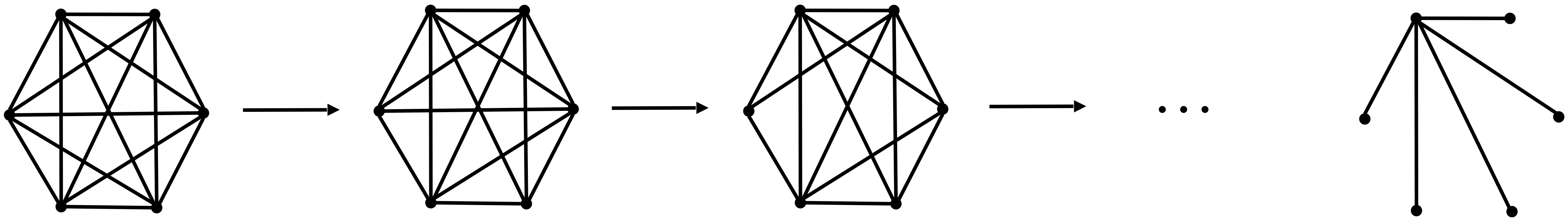

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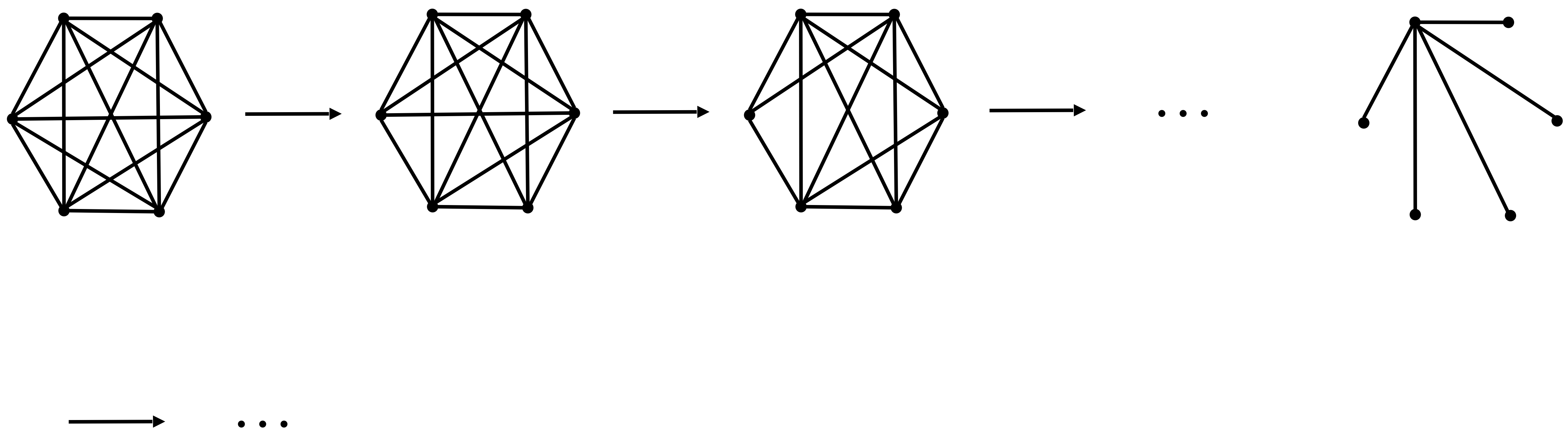

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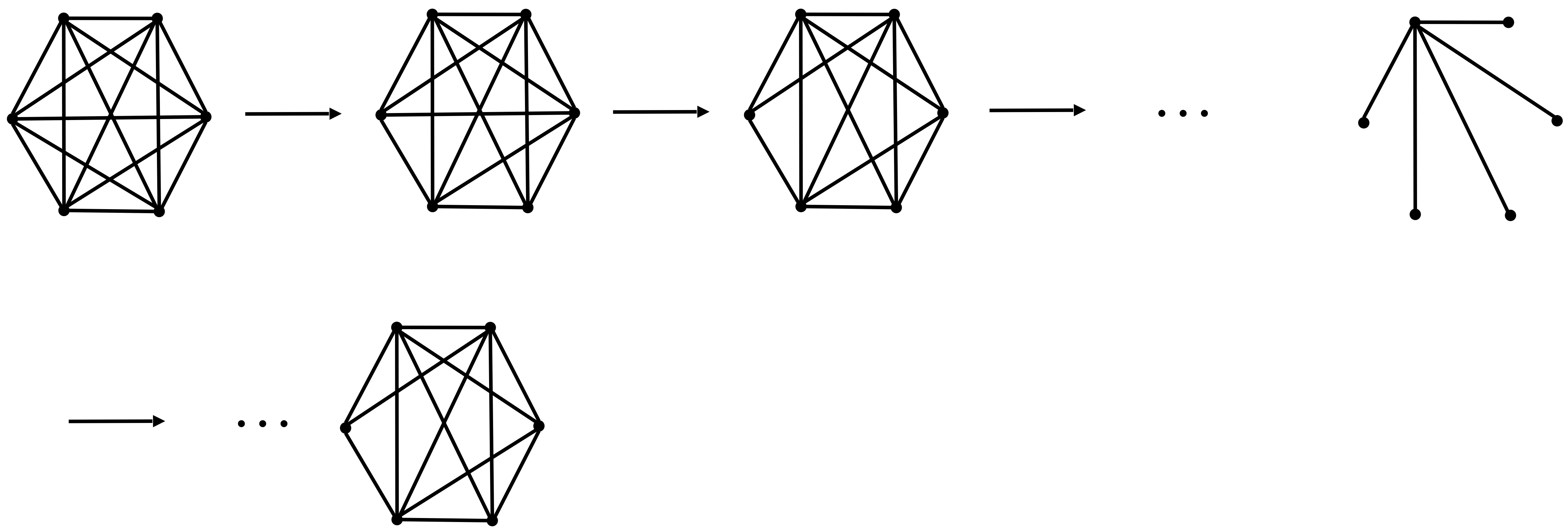

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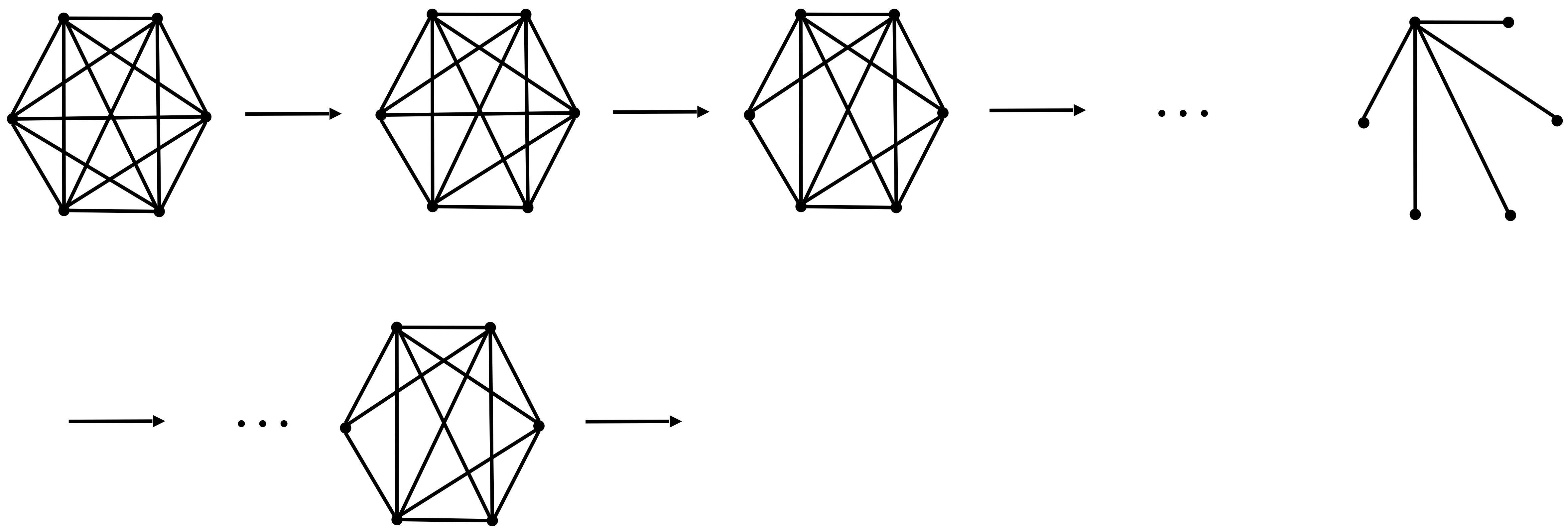

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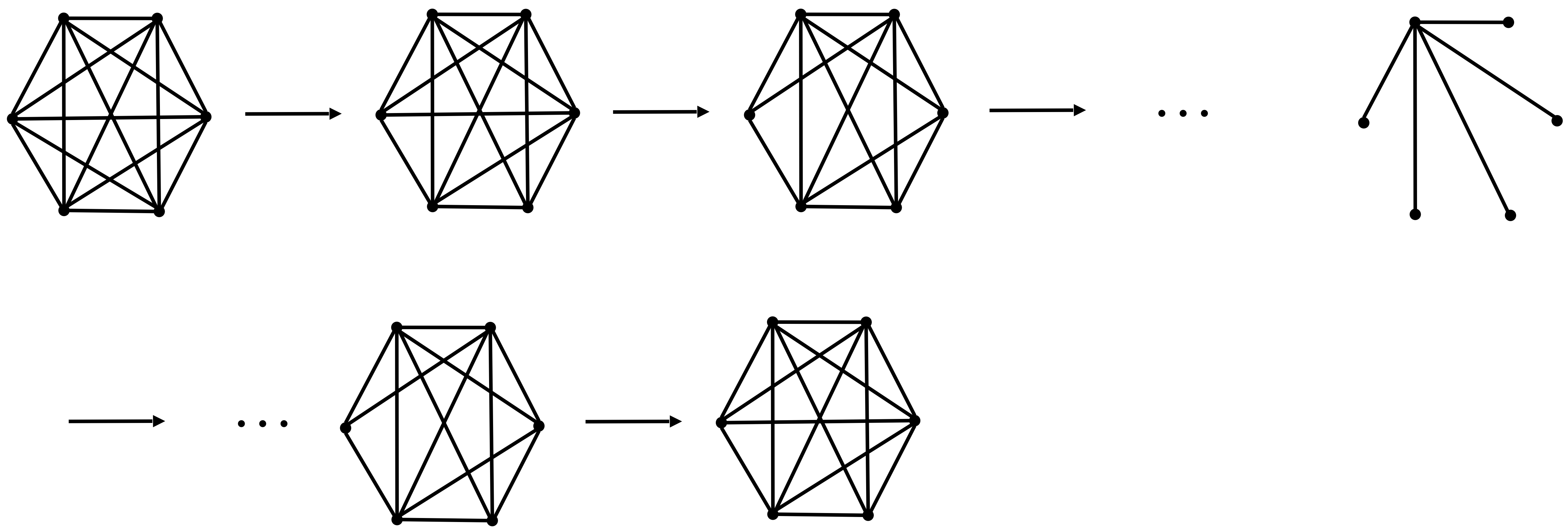

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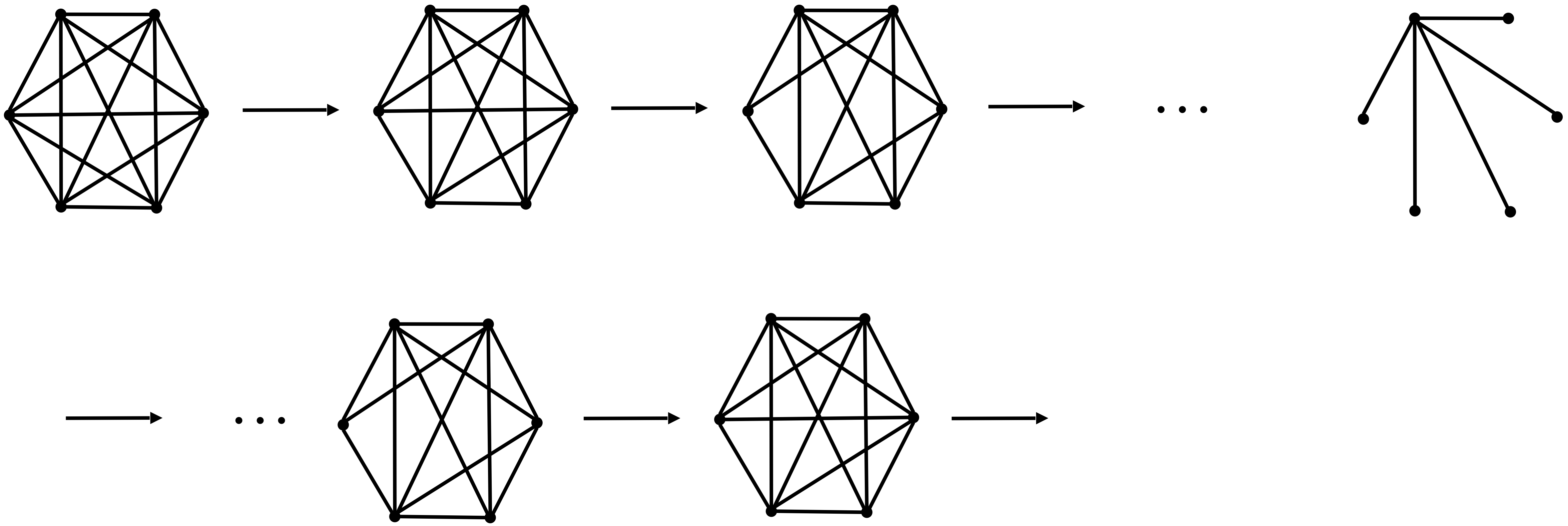

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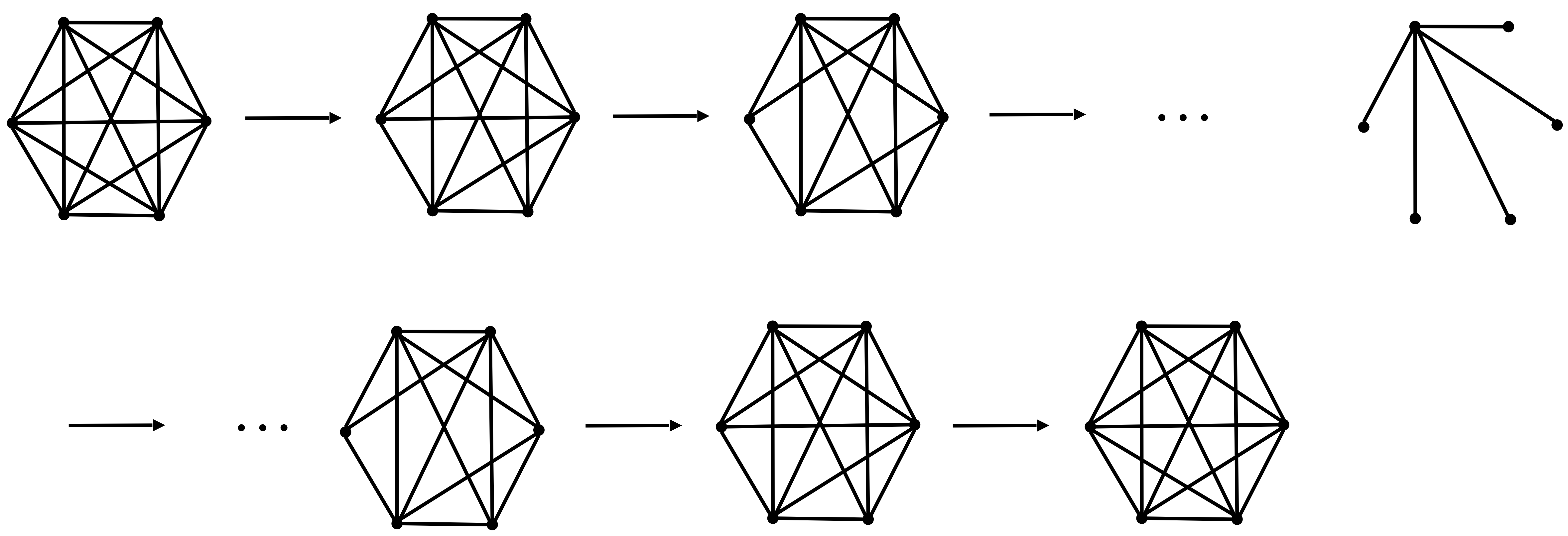

Fault Tolerant Model

Fully dynamic / Dec / Inc model

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Fully dynamic / Dec / Inc model

Too general

Fault Tolerant Model

Fully dynamic / Dec / Inc model

Too general

In many real world networks changes are very limited and transient

Road networks, communication networks etc.

Fault Tolerant Oracle

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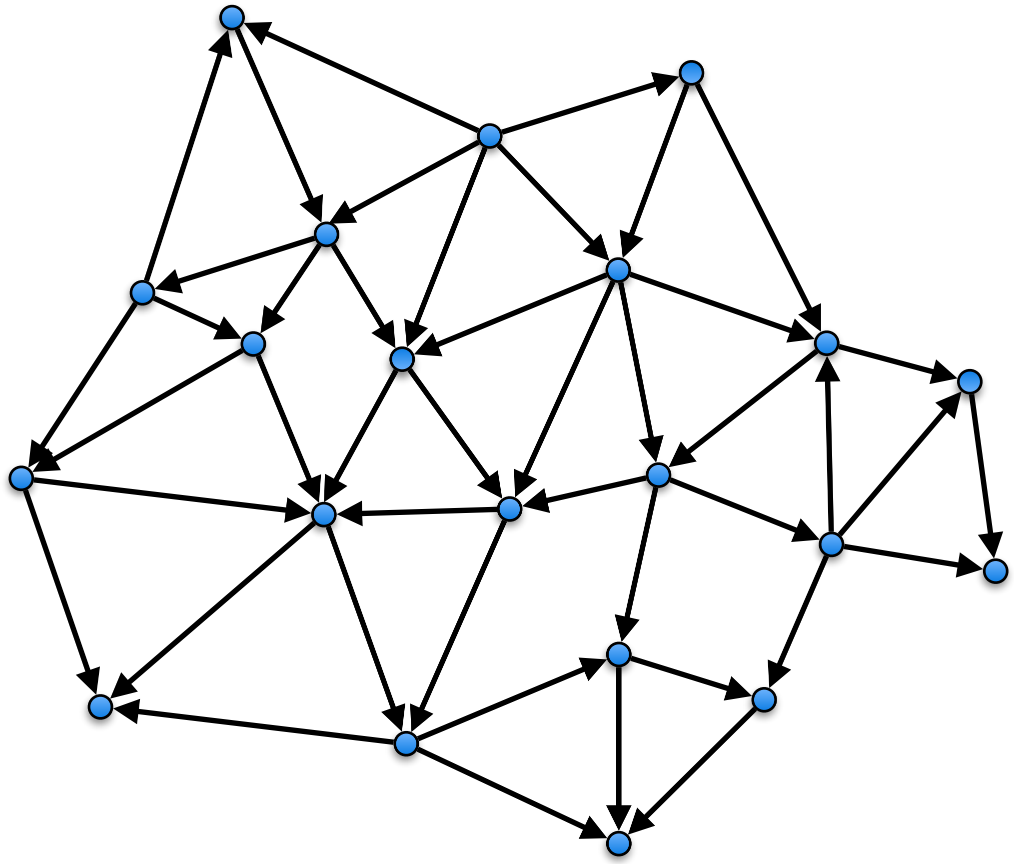
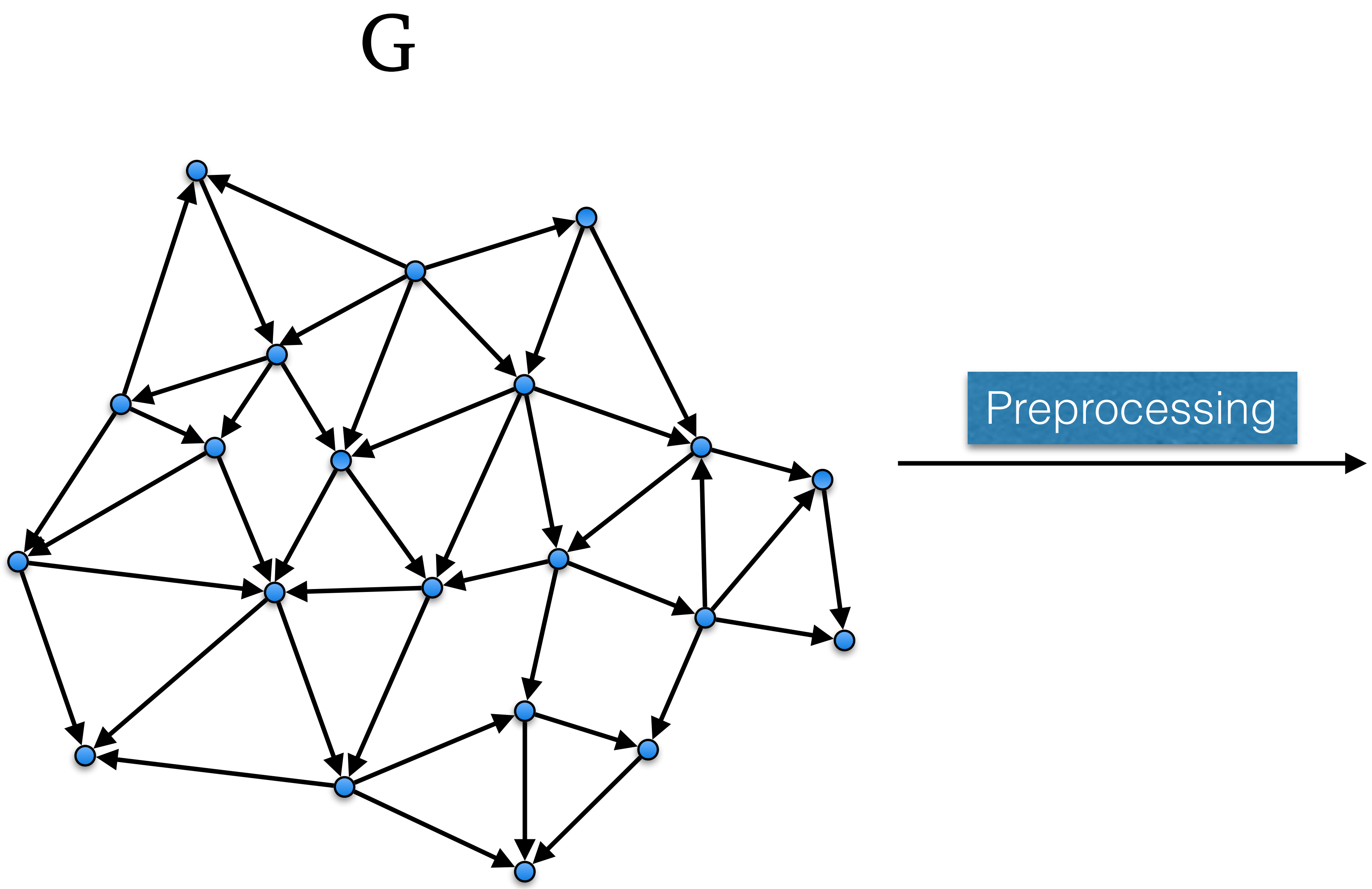
G



Fault Tolerant Oracle



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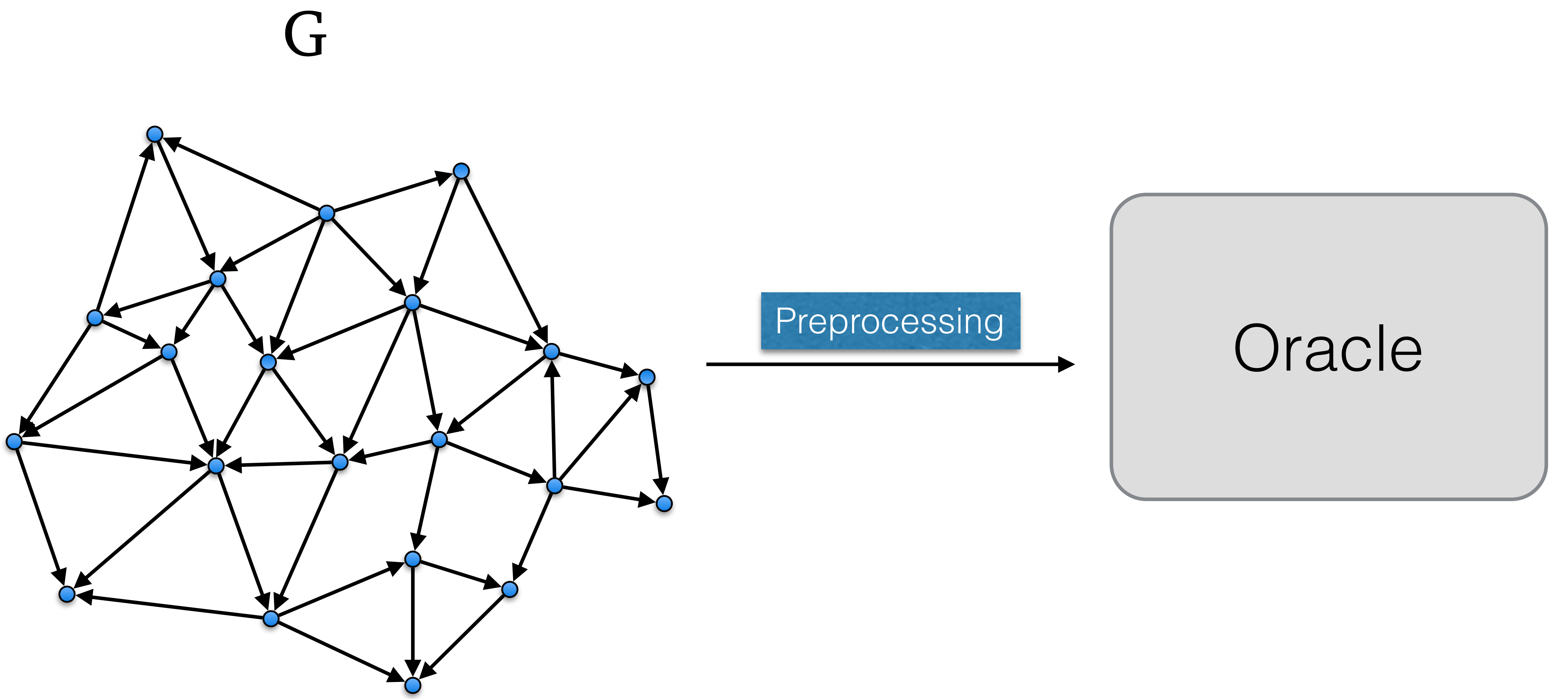

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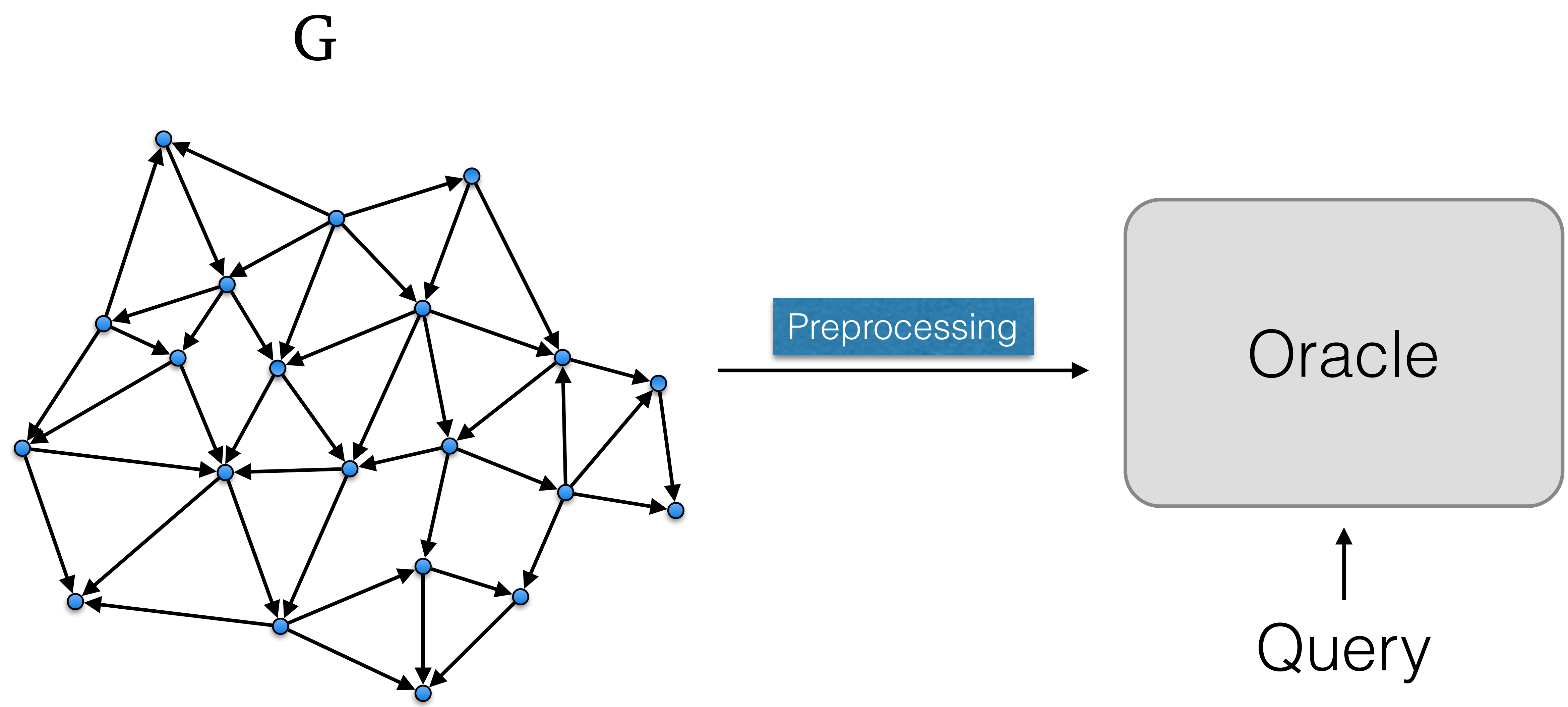

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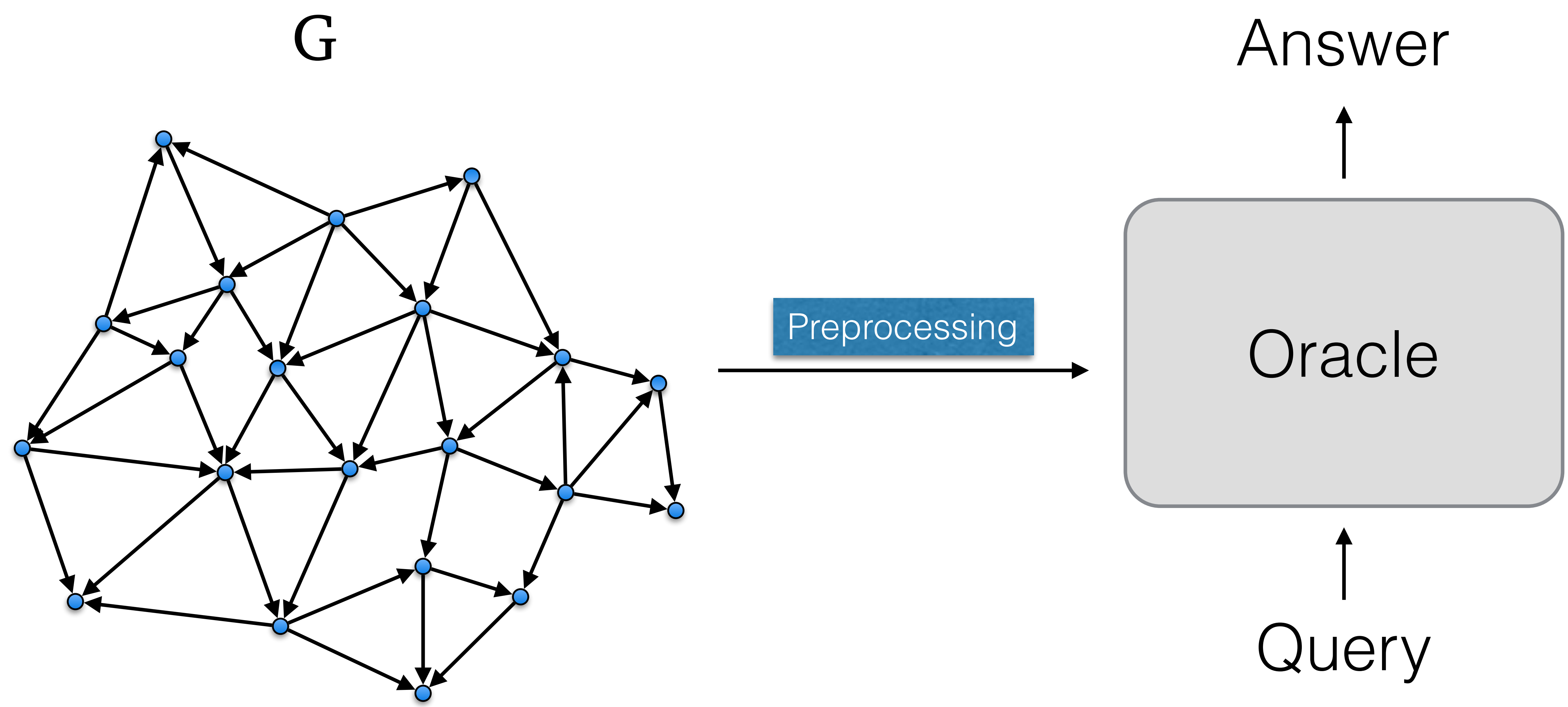

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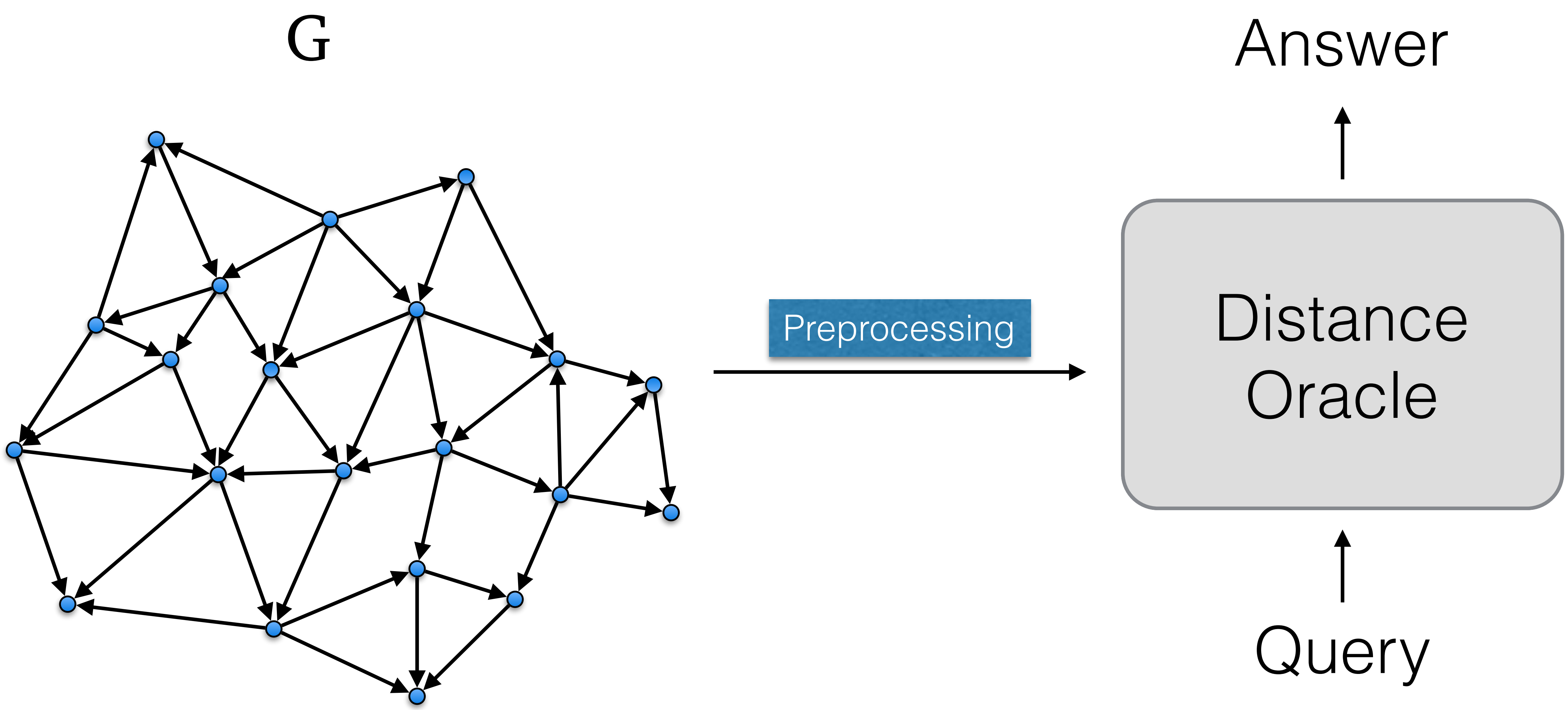

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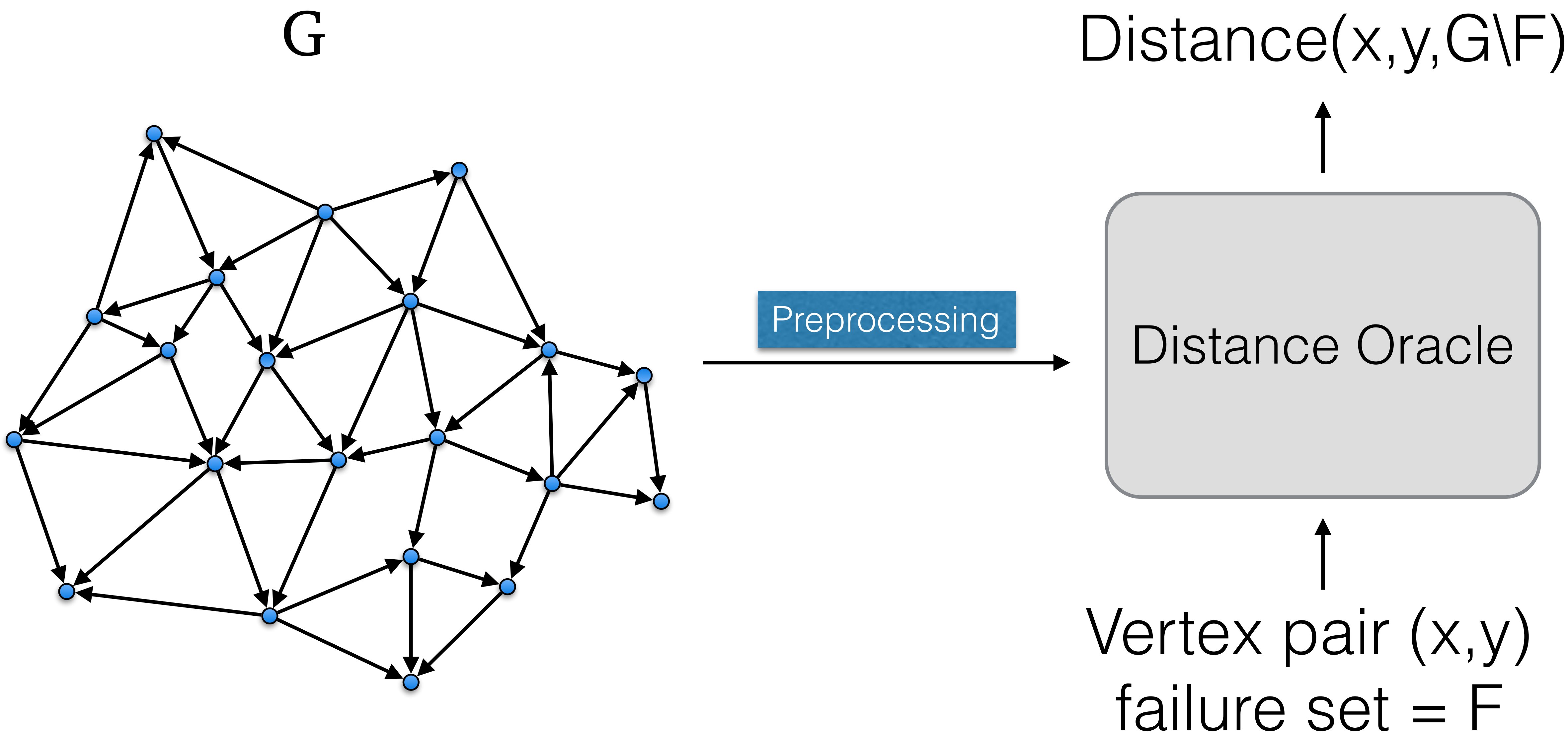

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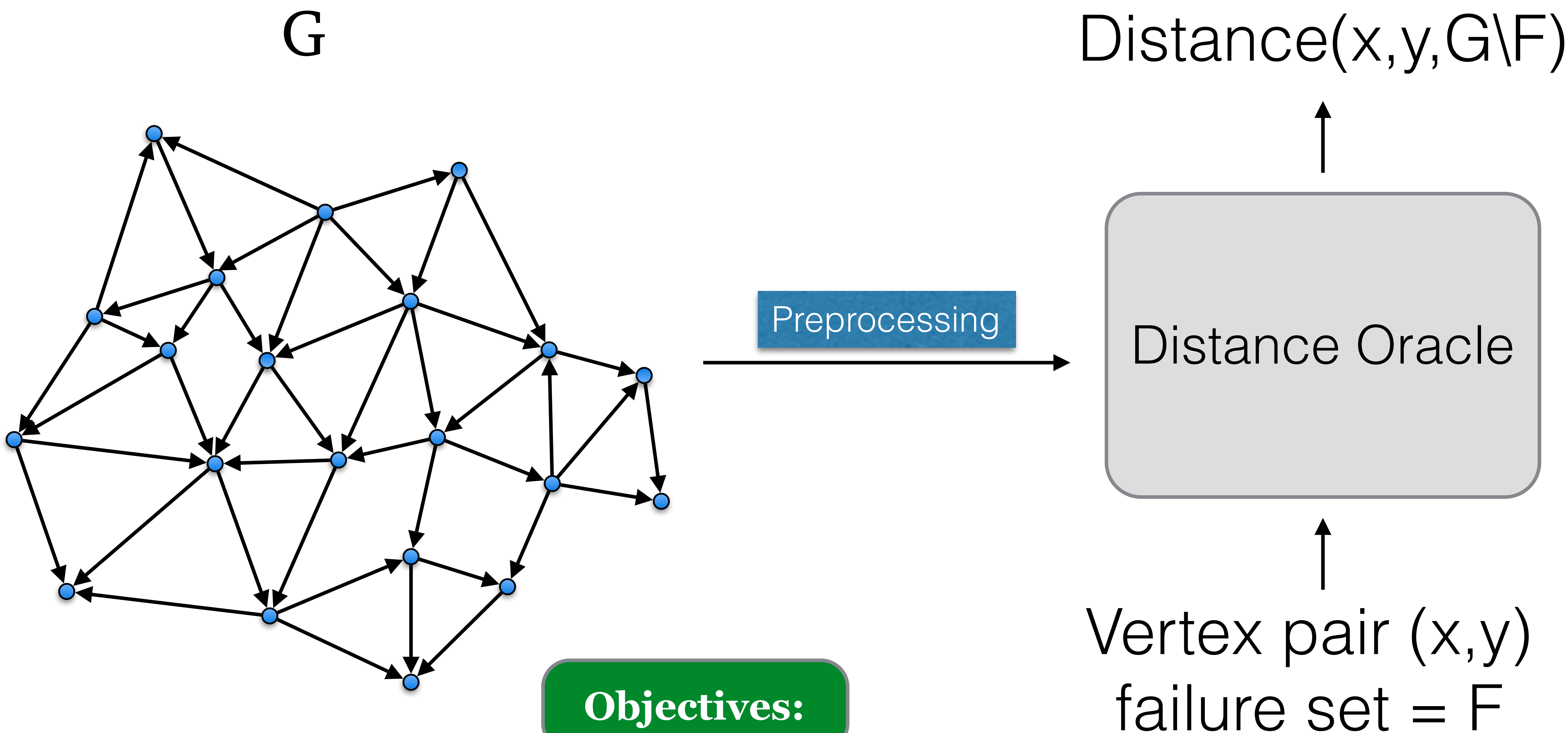
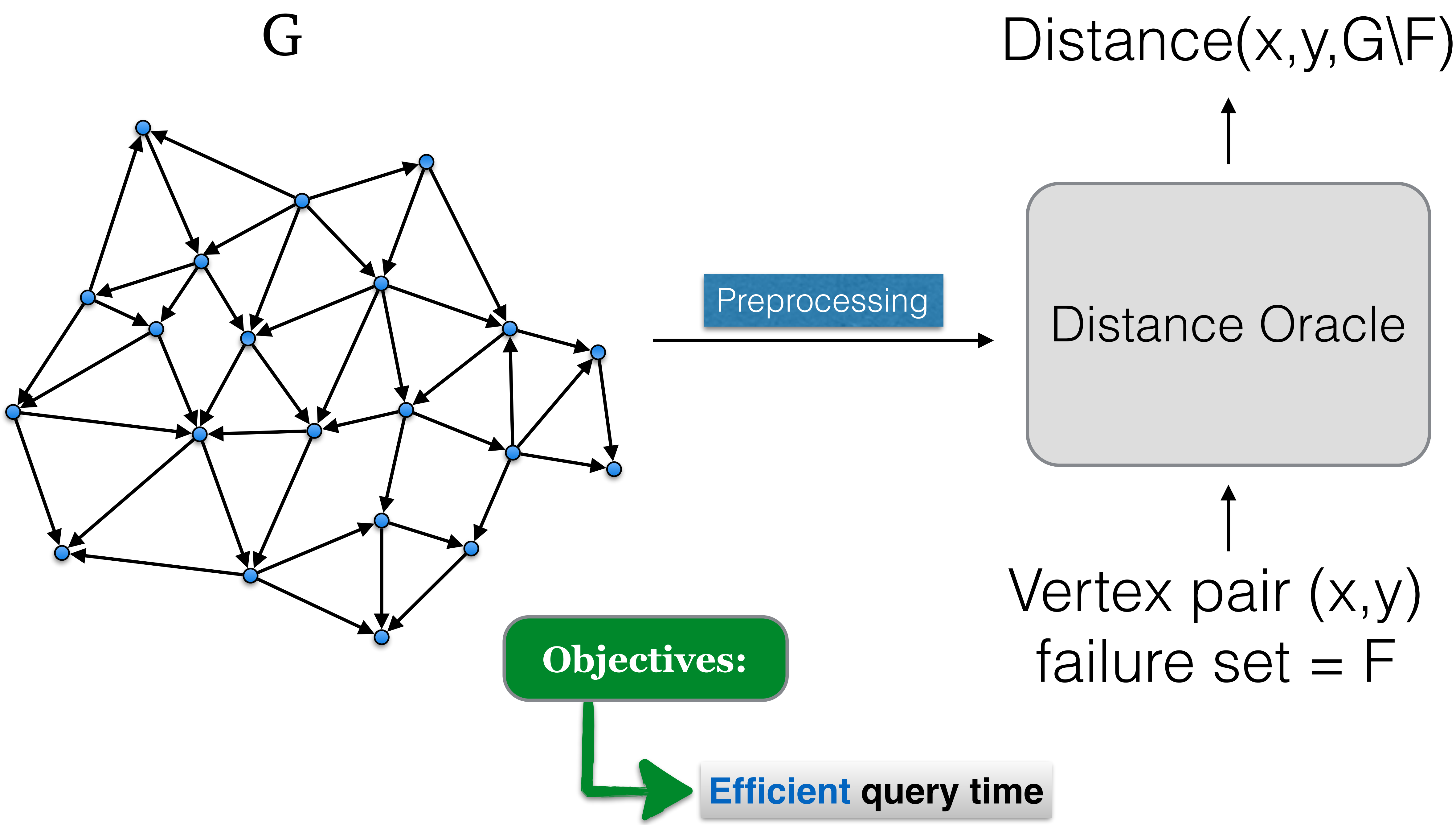


Objectives:

Fault Tolerant Oracle



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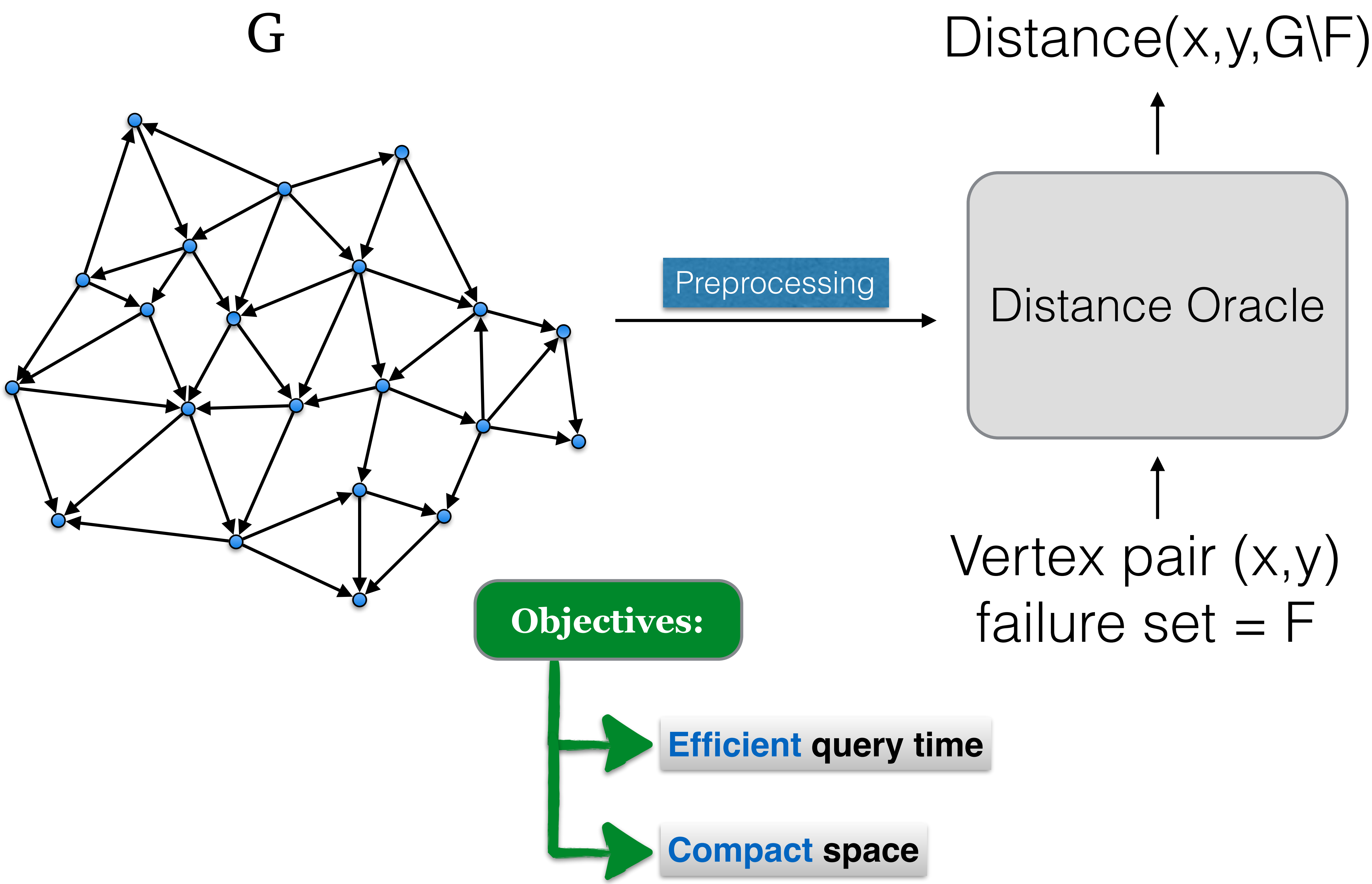

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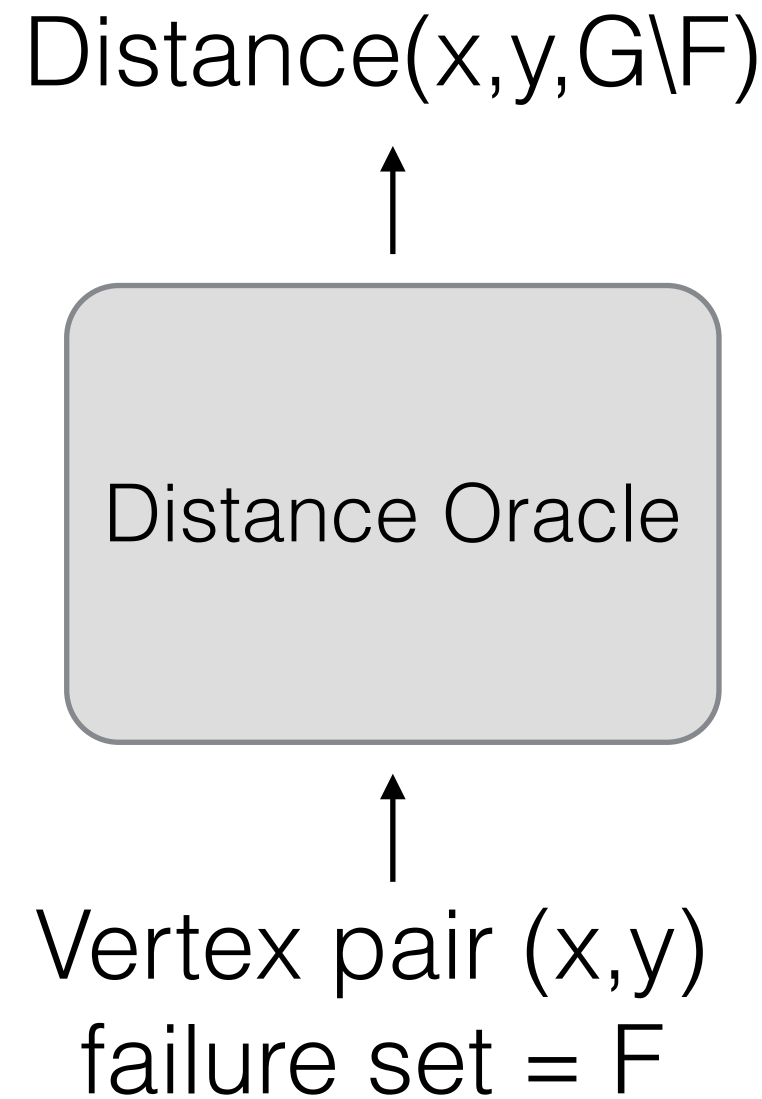
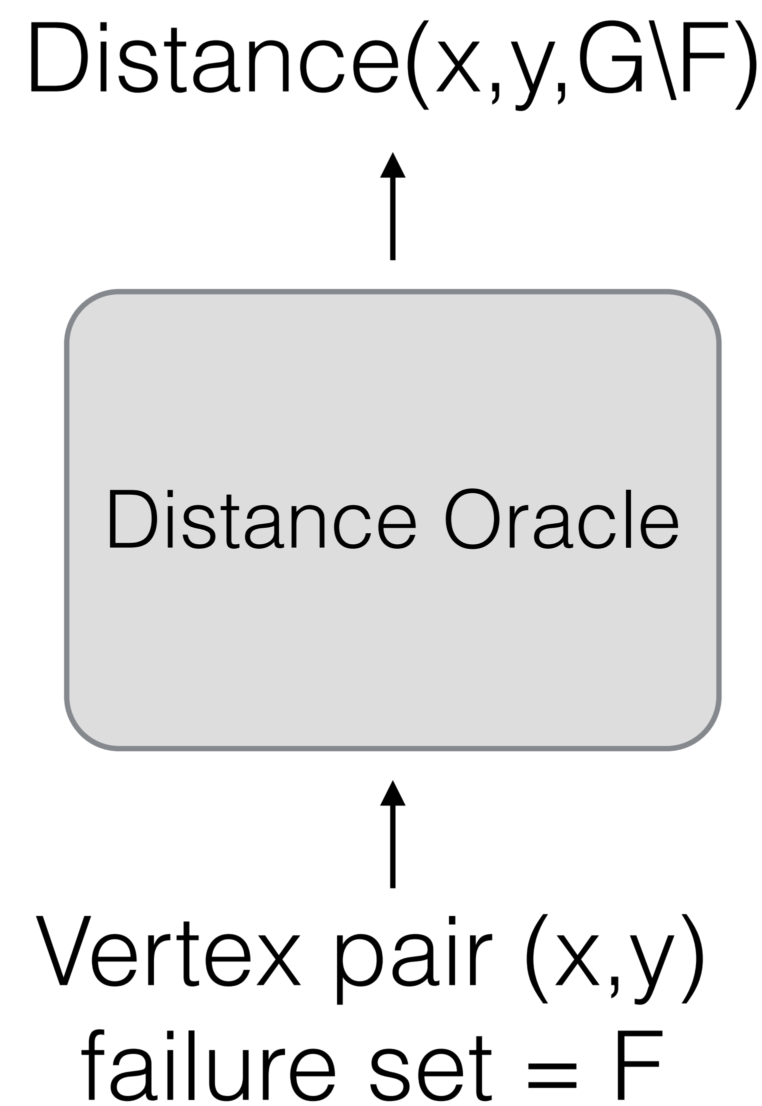


Fault Tolerant Oracle

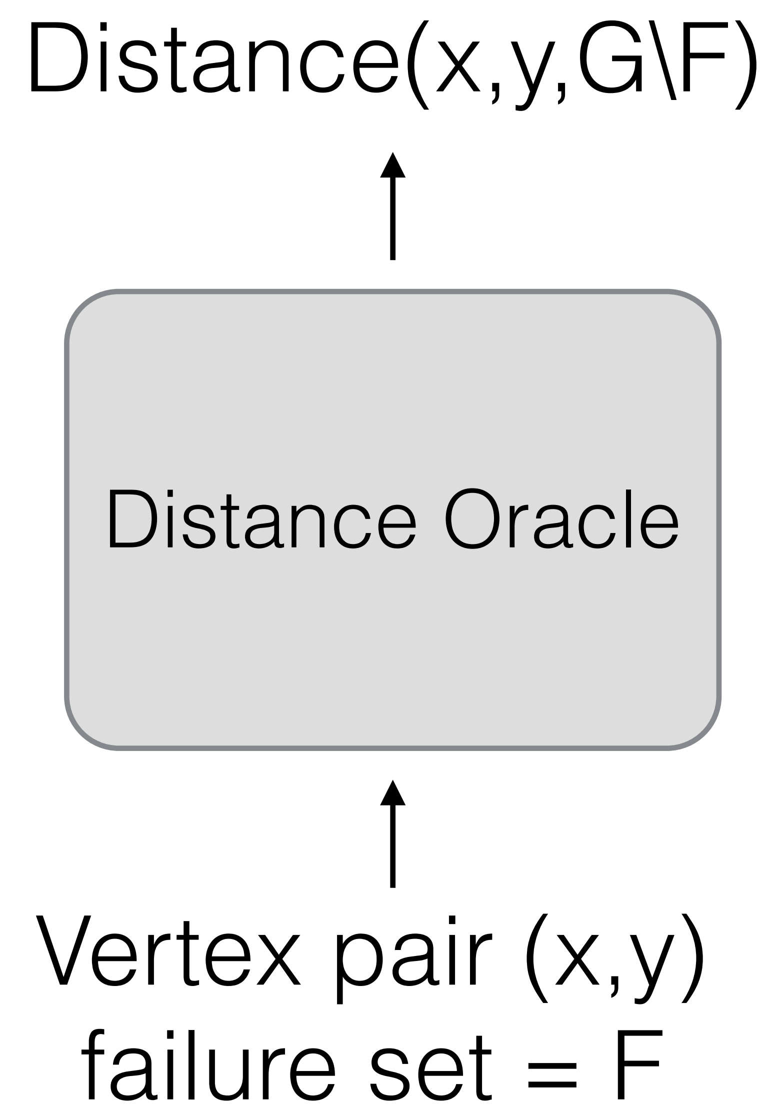
Trivial Solutions:



Fault Tolerant Oracle

Trivial Solutions:

Compute & Store ALL solutions	*Store only graph G*



Fault Tolerant Oracle

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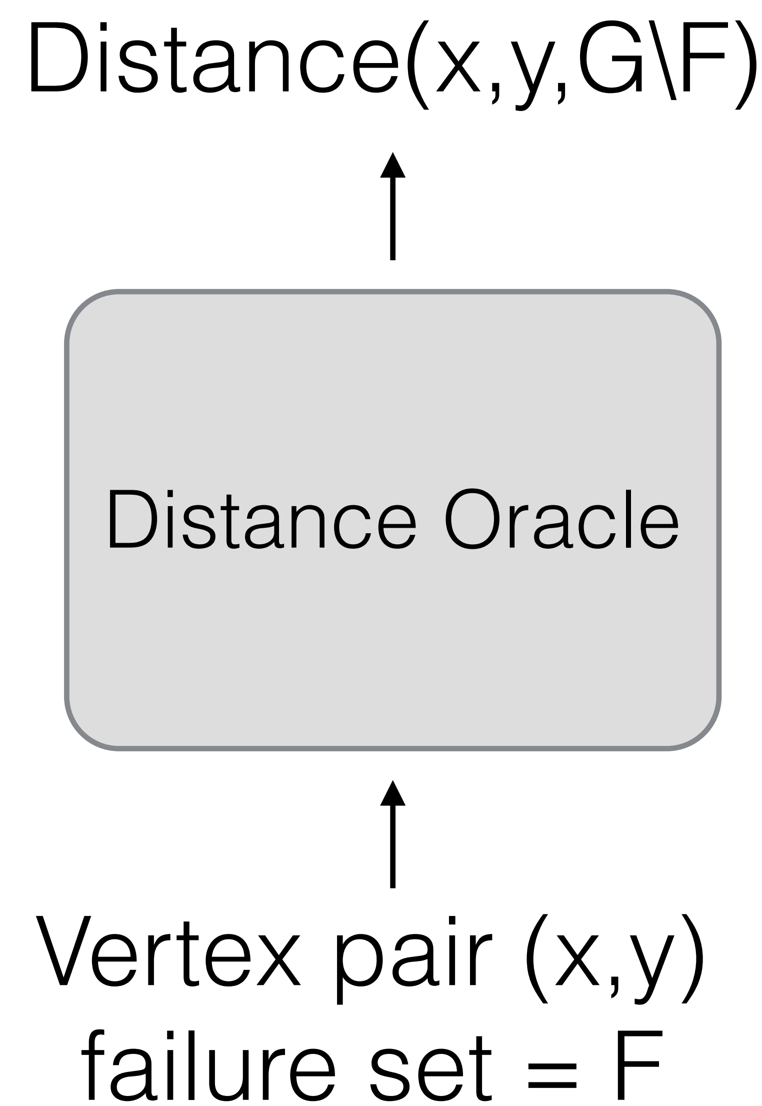
Space = $O({}^{n}C_{k} \cdot n^{2})$ Time = $O(1)$	



Fault Tolerant Oracle

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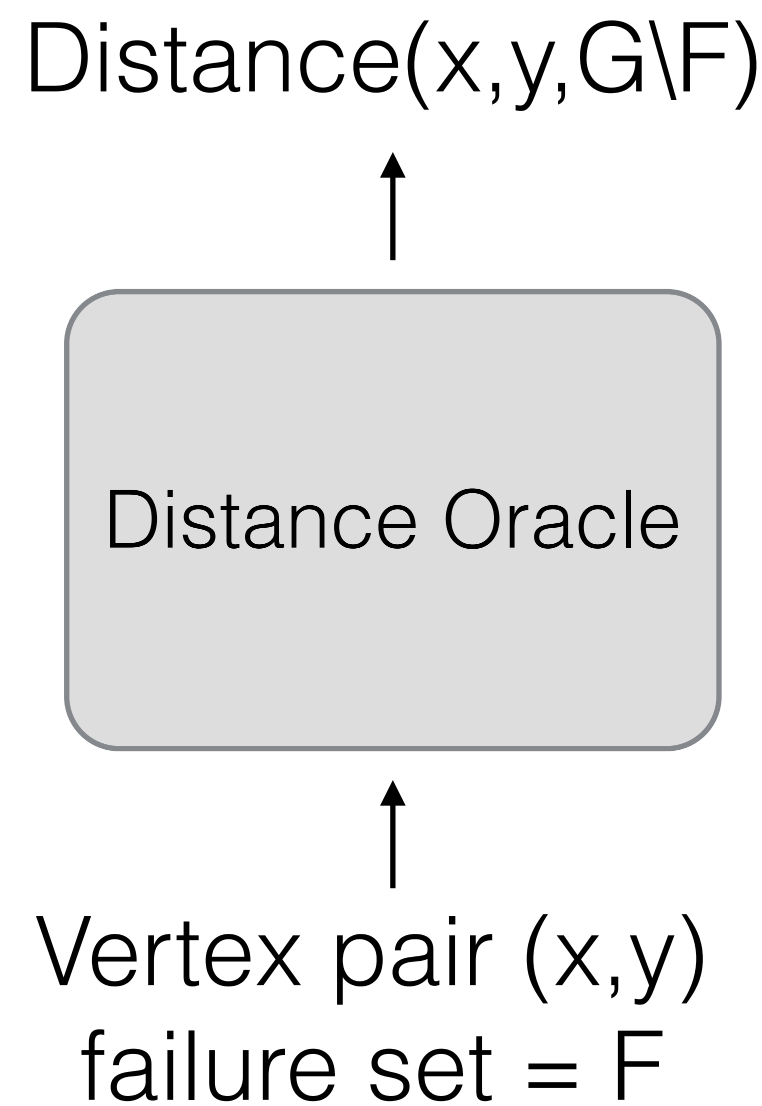

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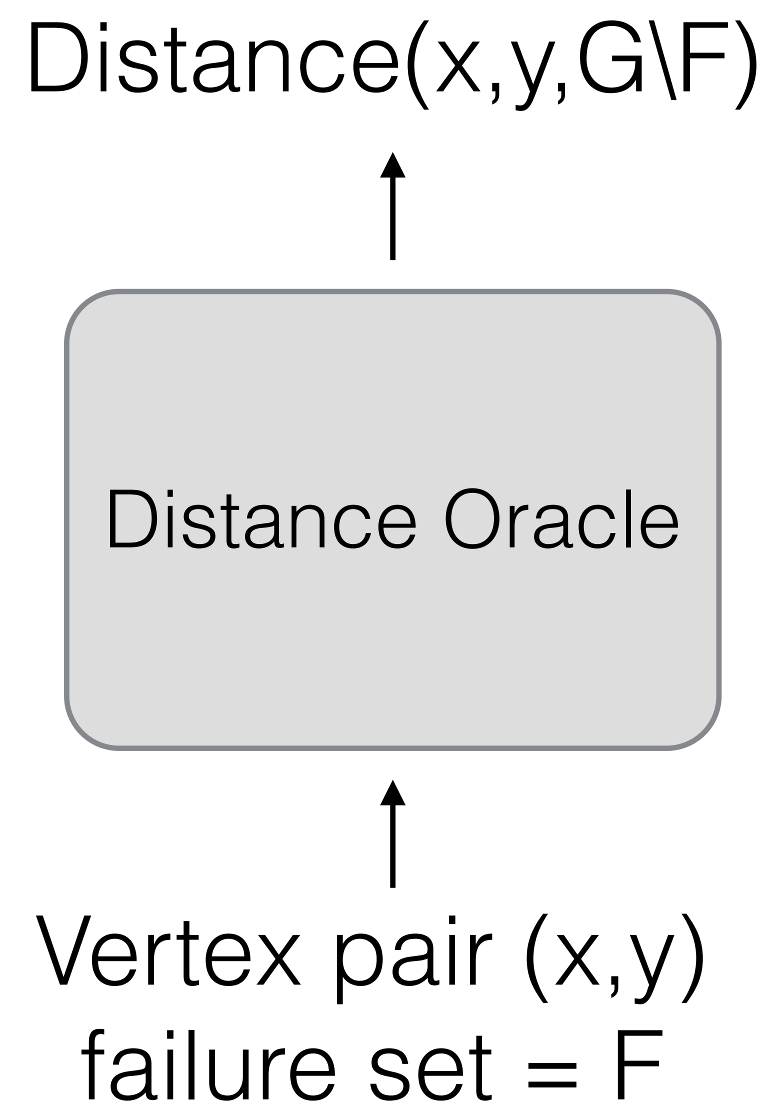
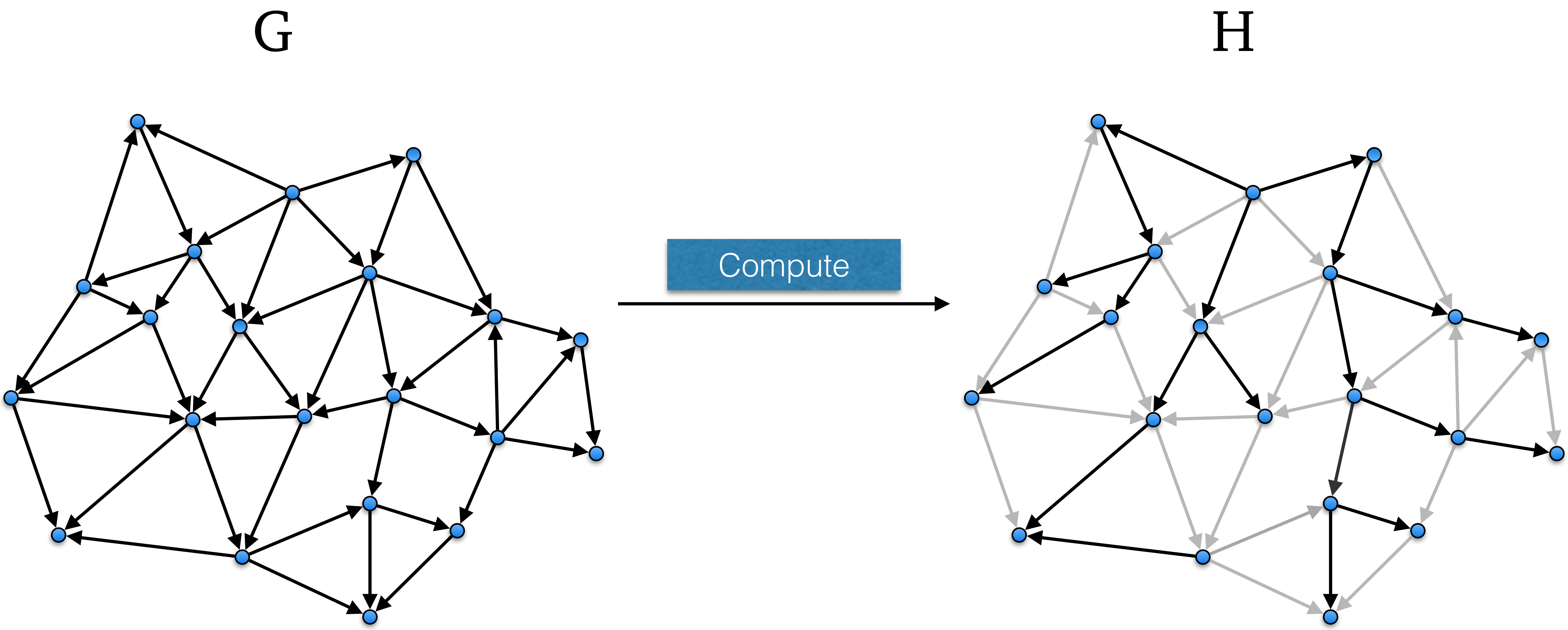


Fault Tolerant Preservers



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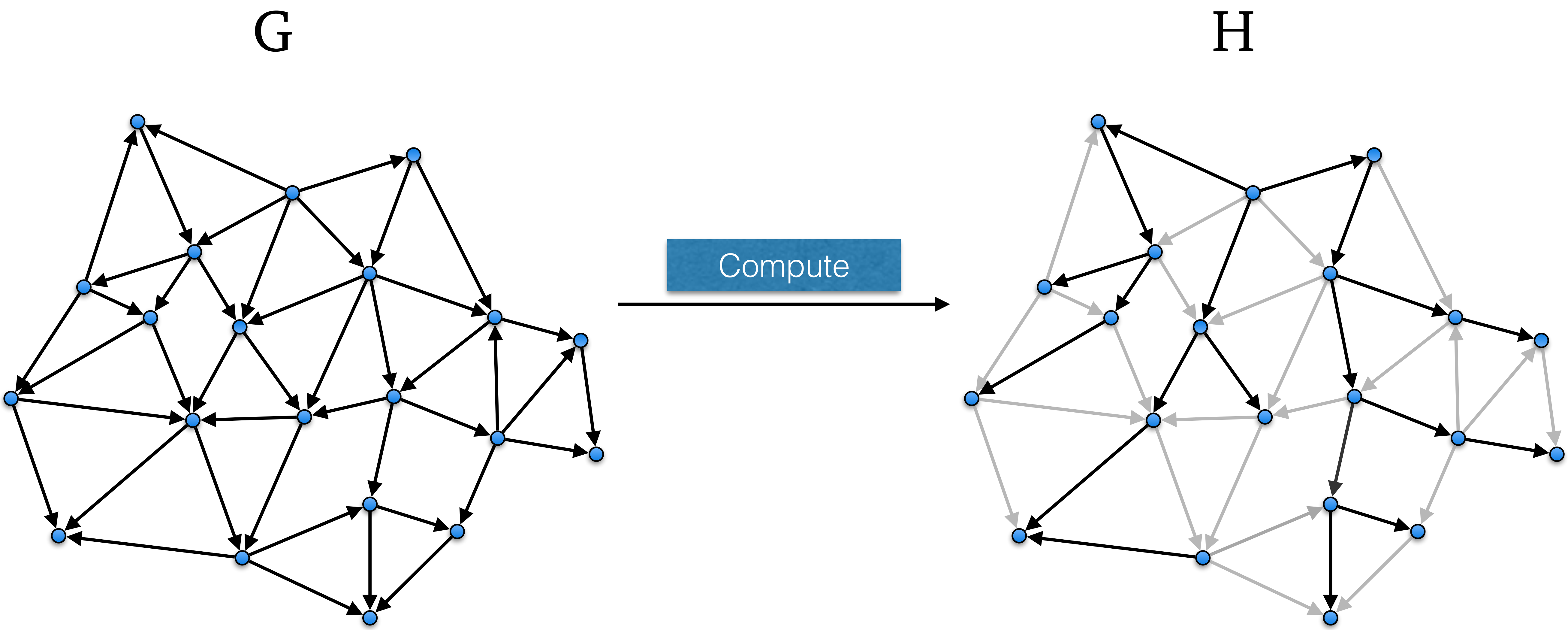

$H \setminus F$ **preserves** a "pre-specified property" of $G \setminus F$,

for all possible F, $|F| \leqslant k$

Previous Works

Many works in the recent decade (partial list):

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Bodwin, Grandoni, Parter, V. William: (ICALP'17) – Distances

This Talk

Problems of Reachability and strong-connectivity:

This Talk

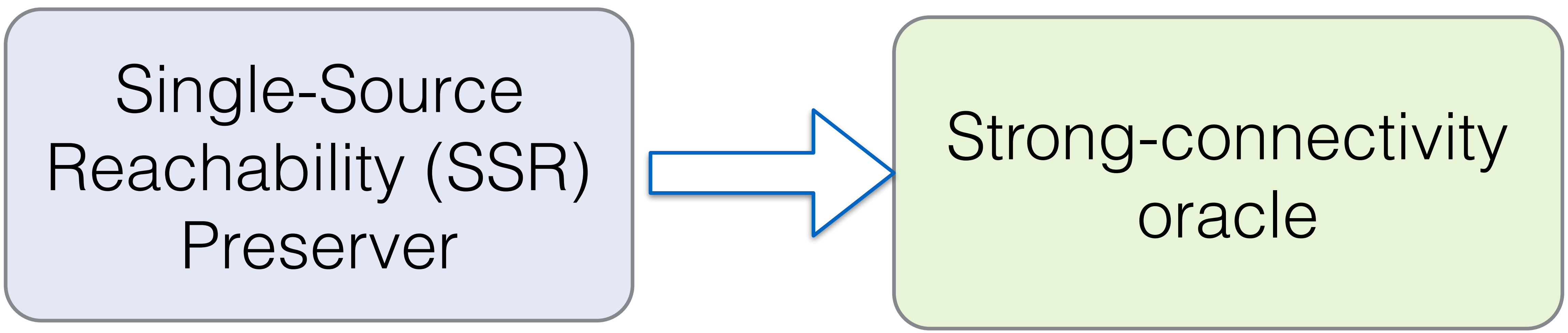
Problems of Reachability and strong-connectivity:

Single-Source Reachability (SSR) Preserver

Problem 1

This Talk

Problems of Reachability and strong-connectivity:



Problem 1

Problem 2

Our Contributions

Problem 1: Reachability Preserver

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Input: directed graph $G=(V,E)$, parameter k, and a source s.

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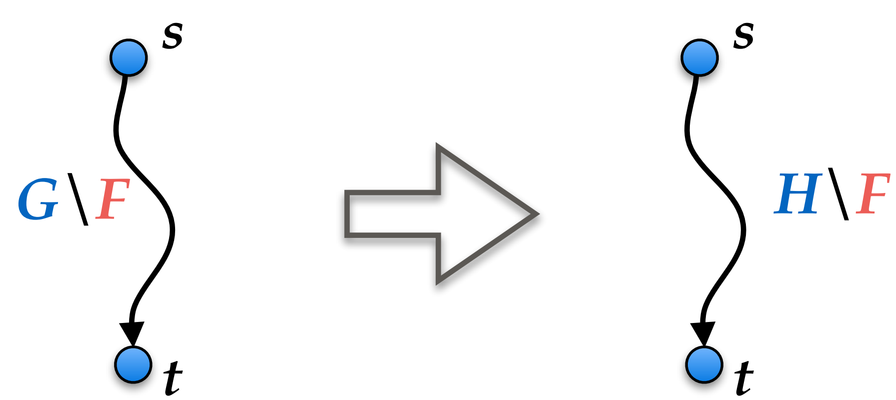

for every $t \in V$

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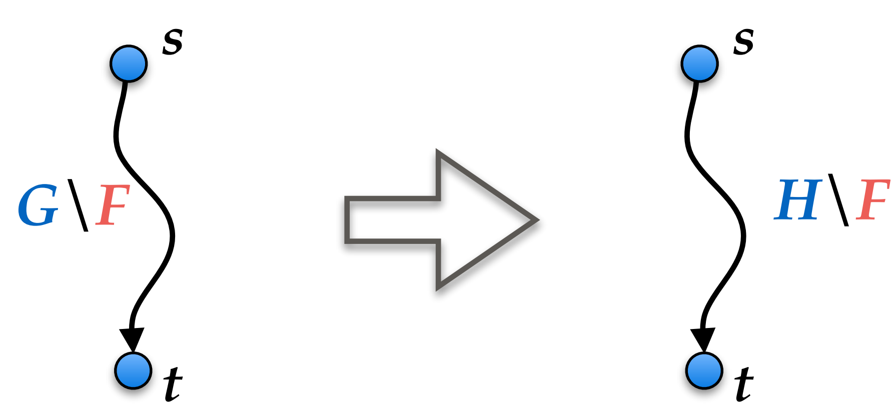

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Example:

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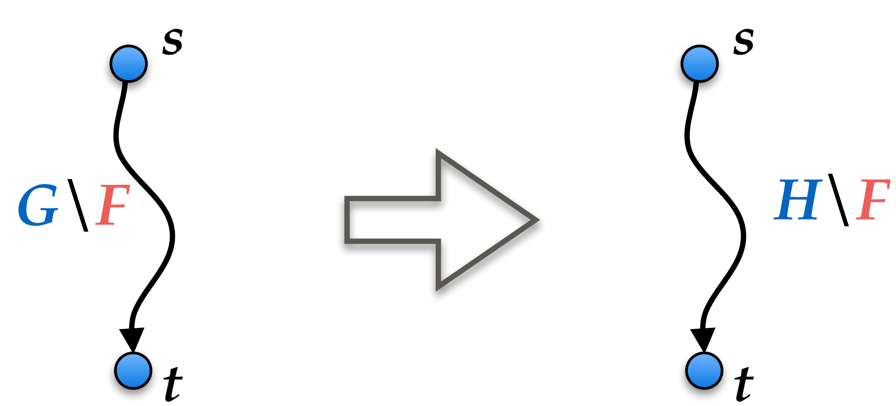

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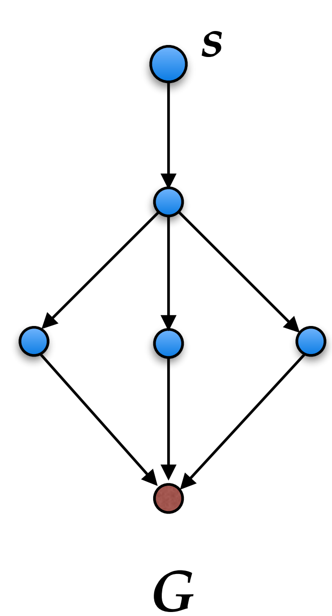

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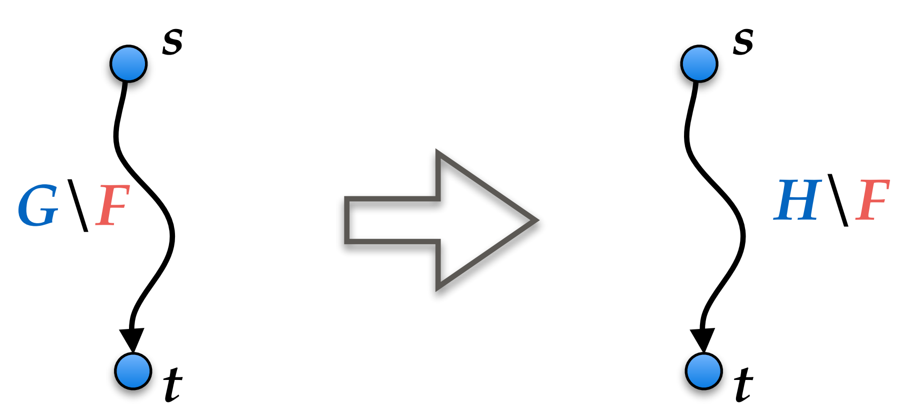

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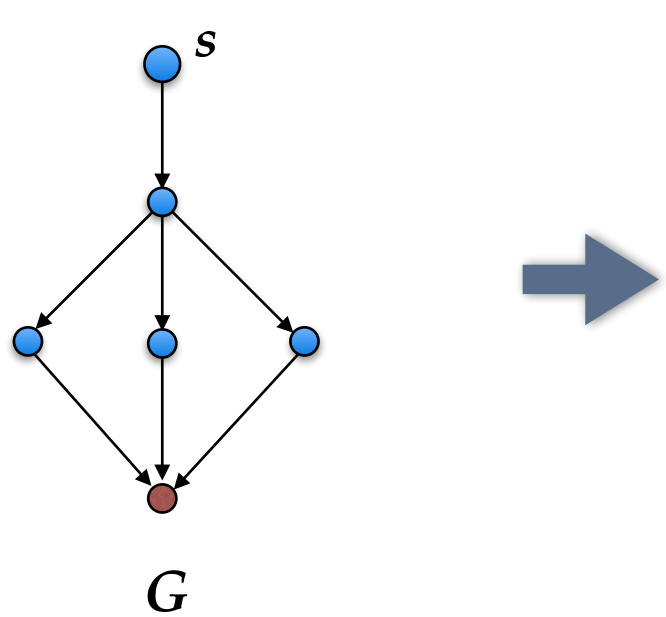

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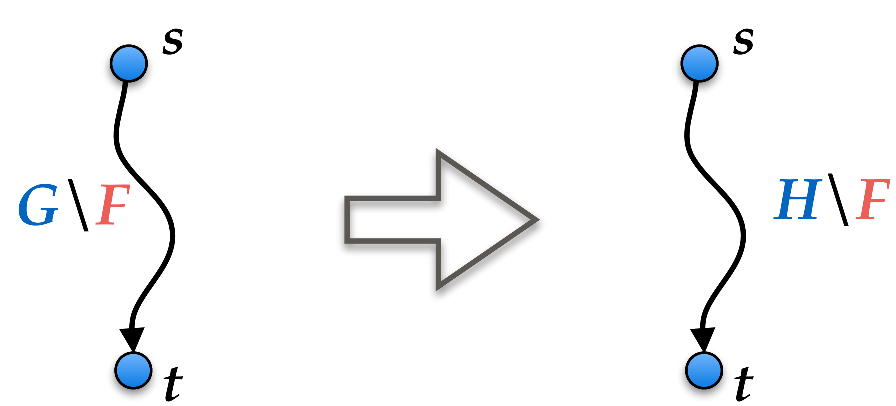

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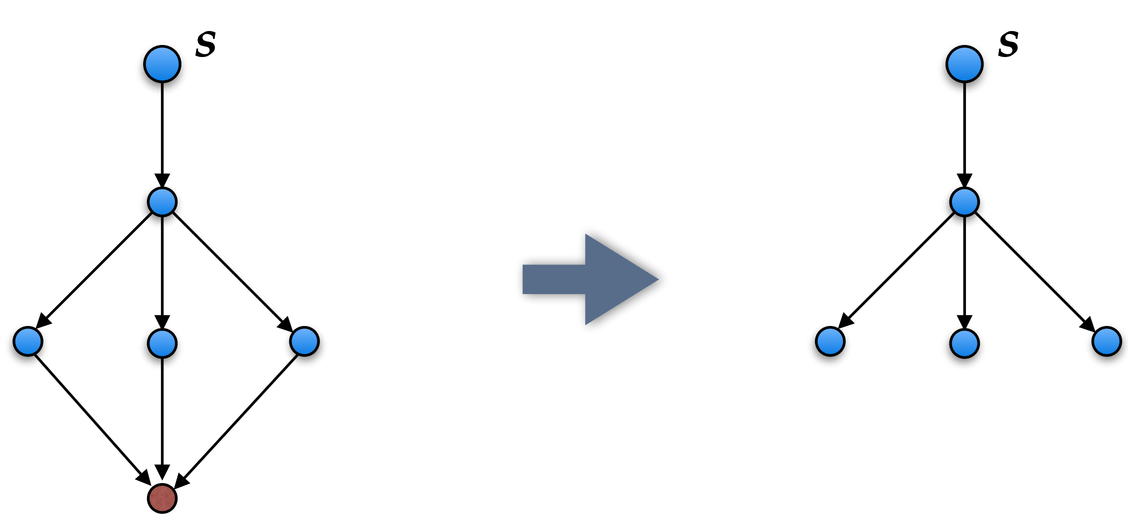

G 1-FT-Preserver

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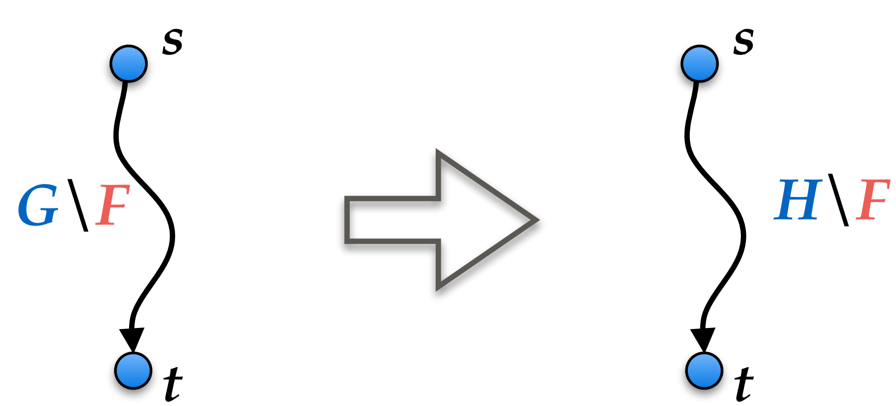
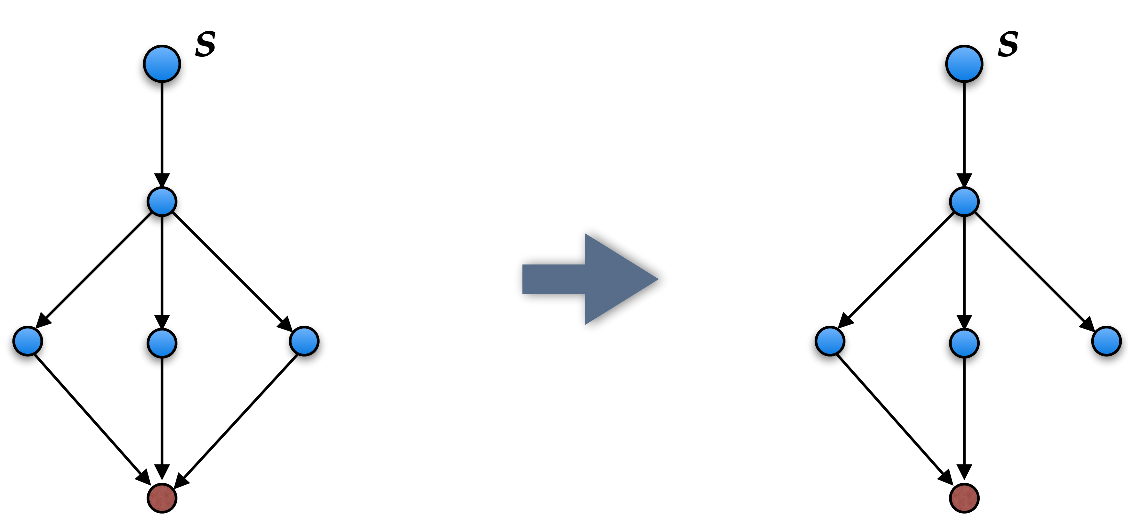


for every $t \in V$

Example:

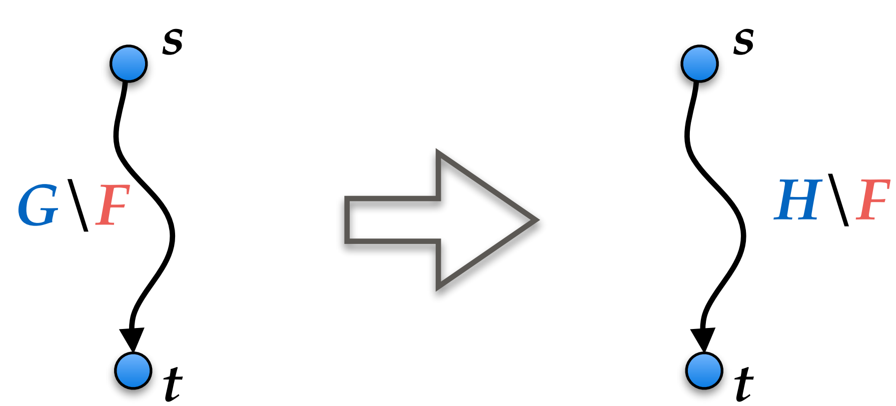


G **1**-FT-Preserver

Problem 1: Reachability Preserver

Input: directed graph $G=(V,E)$, parameter k, and a source s.

Output: a sparse subgraph H of G that on any set F of k edges satisfies:



for every $t \in V$

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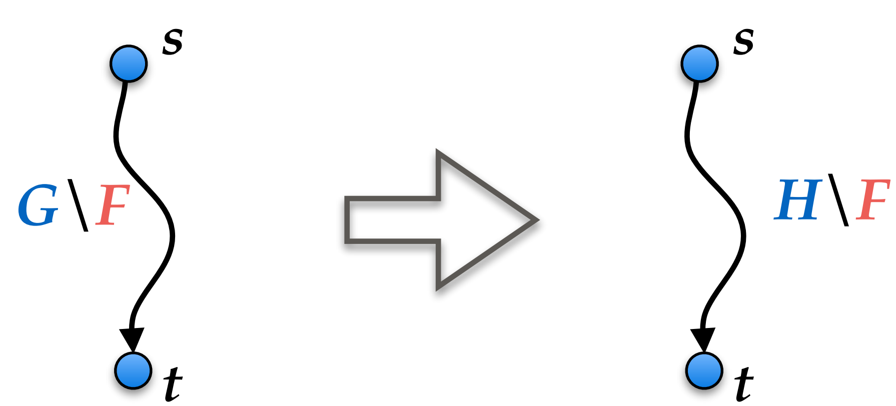

for every $t \in V$

Prior Work:

[Lengauer and Tarjan (1979)]:

- k=1 (single failure)
- An upper bound of $(2n)$

Problem 1: Reachability Preserver

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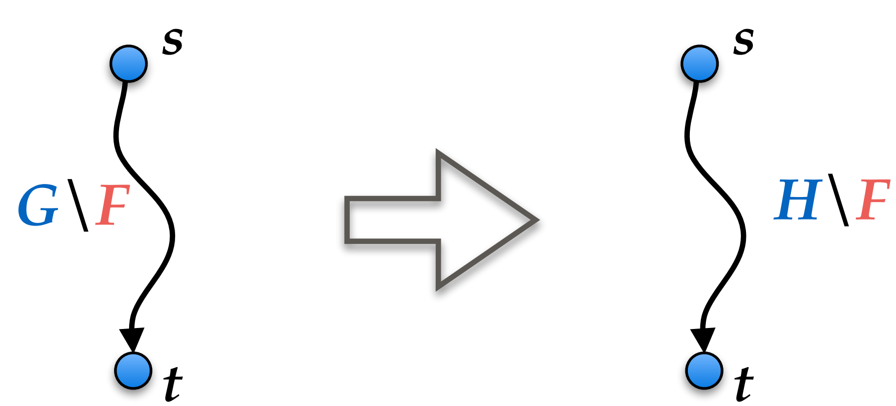

for every $t \in V$

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Our Results for general k:

Problem 1: Reachability Preserver

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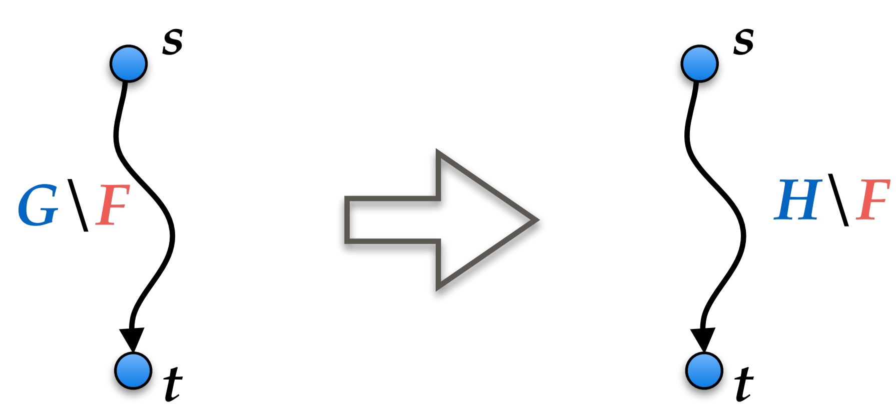

for every $t \in V$

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Our Results for general k:

Upper Bound:

Problem 1: Reachability Preserver

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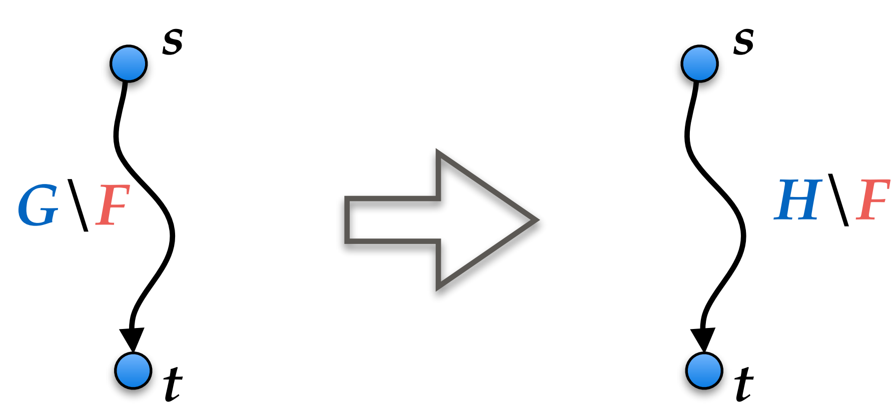

for every $t \in V$

Prior Work:

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- k=1 (single failure)
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Our Results for general k:

Upper Bound: $O(2^k n)$ edges

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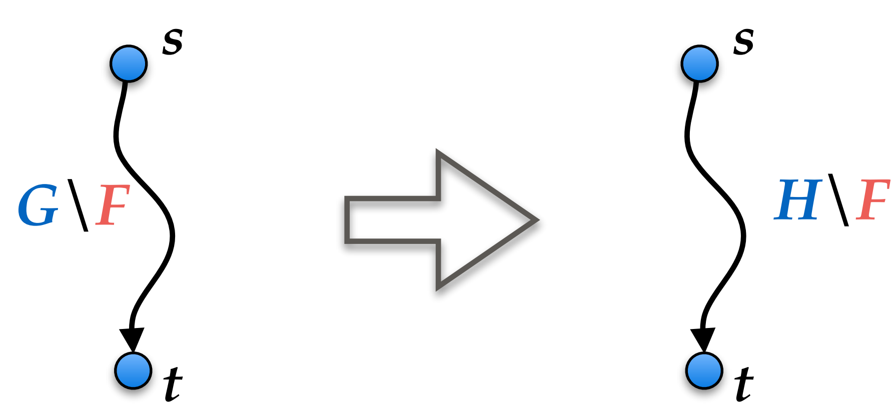

for every $t \in V$

Prior Work:

[Lengauer and Tarjan (1979)]:

- k=1 (single failure)
- An upper bound of $(2n)$

Our Results for general k:

Upper Bound: $O(2^k n)$ edges

Lower Bound:

Problem 1: Reachability Preserver

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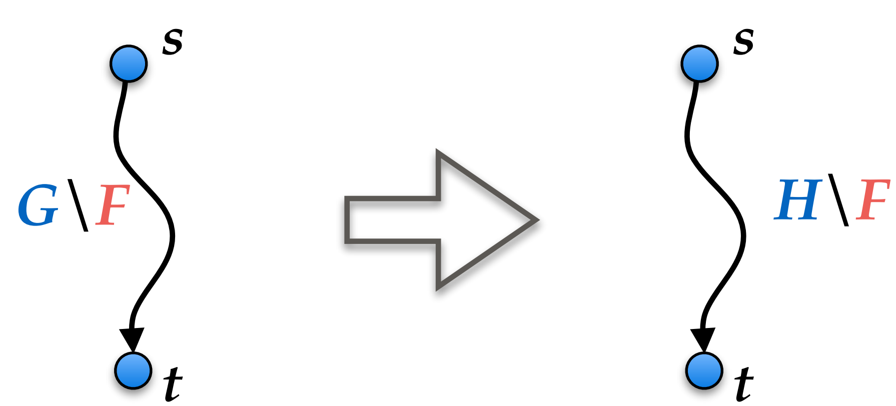
Output: a sparse subgraph H of G that on any set F of k edges satisfies:



for every $t \in V$

Prior Work:

[Lengauer and Tarjan (1979)]:

- k=1 (single failure)
- An upper bound of $(2n)$

Our Results for general k:

Upper Bound: $O(2^k n)$ edges

Lower Bound: Existential bound of $\Omega(2^k n)$ edges

Problem 1: Reachability Preserver

Implication (i):

Problem 1: Reachability Preserver

Implication (i):

Reachability Oracle

Problem 1: Reachability Preserver

Implication (i):

Problem 1: Reachability Preserver

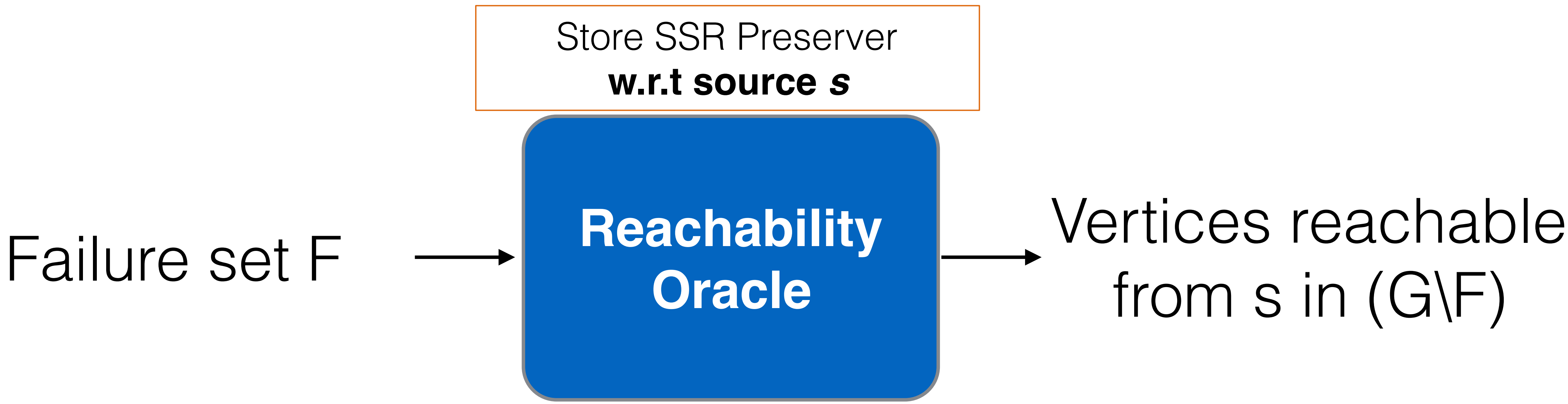
Implication (i):

Store SSR Preserver
w.r.t source *s*

Failure set F → **Reachability Oracle** → Vertices reachable from s in (G\F)

Problem 1: Reachability Preserver

Implication (i):



Time: $O(2^k n)$ time!

Problem 1: Reachability Preserver

Implication (i):

Store SSR Preserver
w.r.t source *s*

Failure set F → **Reachability Oracle** → Vertices reachable from s in (G\F)

Time: O(2^k n) time!

Space: O(2^k n)

Problem 1: Reachability Preserver

Implication (ii):

Problem 1: Reachability Preserver

Implication (ii):

SCC
Oracle

Problem 1: Reachability Preserver

Implication (ii):

Problem 1: Reachability Preserver

Implication (ii):

Store **ALL** SSR Preservers
w.r.t. G as well G-reverse

Vertex y, set F → **SCC Oracle** → SCC of y in (G\F)

Problem 1: Reachability Preserver

Implication (ii):

Store **ALL** SSR Preservers
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Vertex y, set F → **SCC Oracle** → SCC of y in (G\F)

Time: O($2^k n$) time!

Problem 1: Reachability Preserver

Implication (ii):

Store **ALL** SSR Preservers w.r.t. G as well G-reverse

Vertex y, set F → **SCC Oracle** → SCC of y in (G\F)

Time: $O(2^k n)$ time!

Space: $O(2^k n^2)$

Problem 1: Reachability Preserver

Proof Snippet

Problem 1: Reachability Preserver

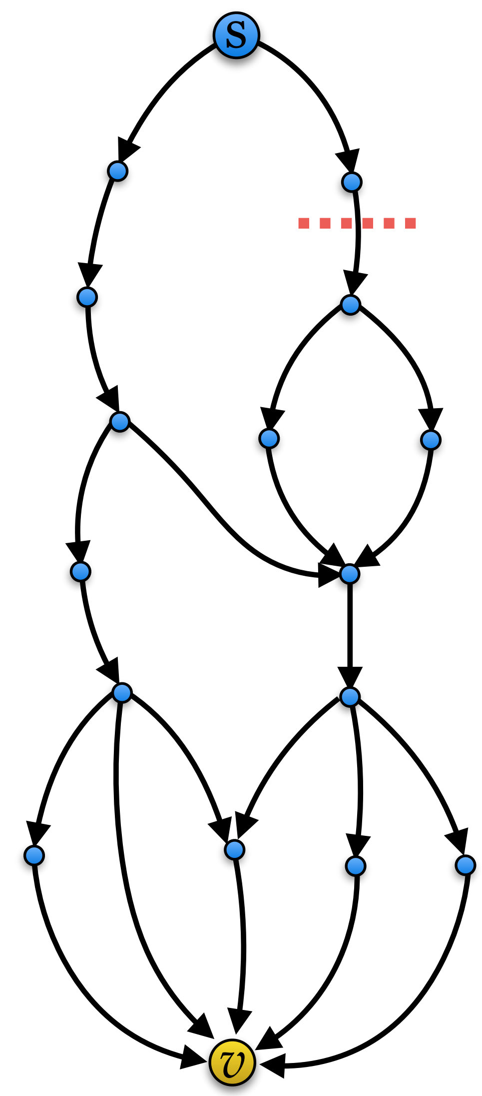
Proof Snippet

Farthest min-cut

Problem 1: Reachability Preserver

Proof Snippet

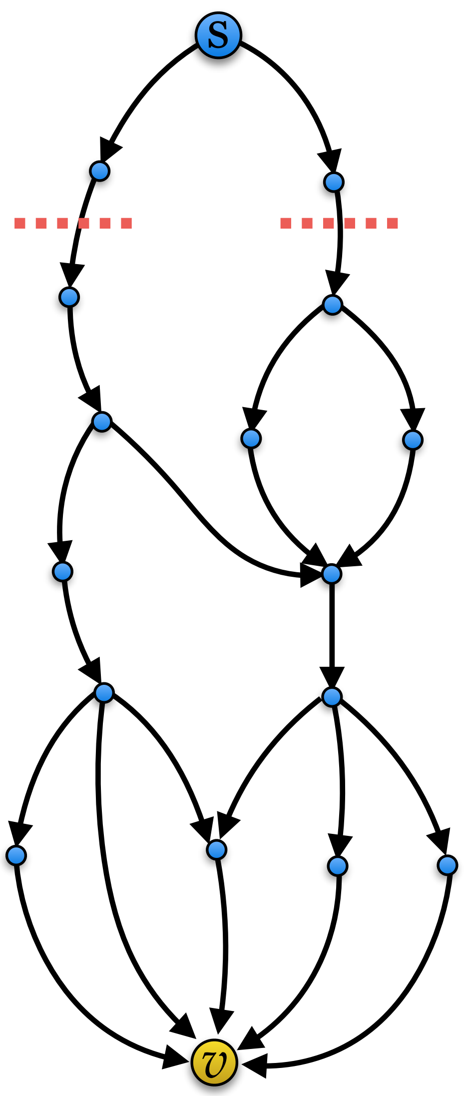
Farthest min-cut



Problem 1: Reachability Preserver

Proof Snippet

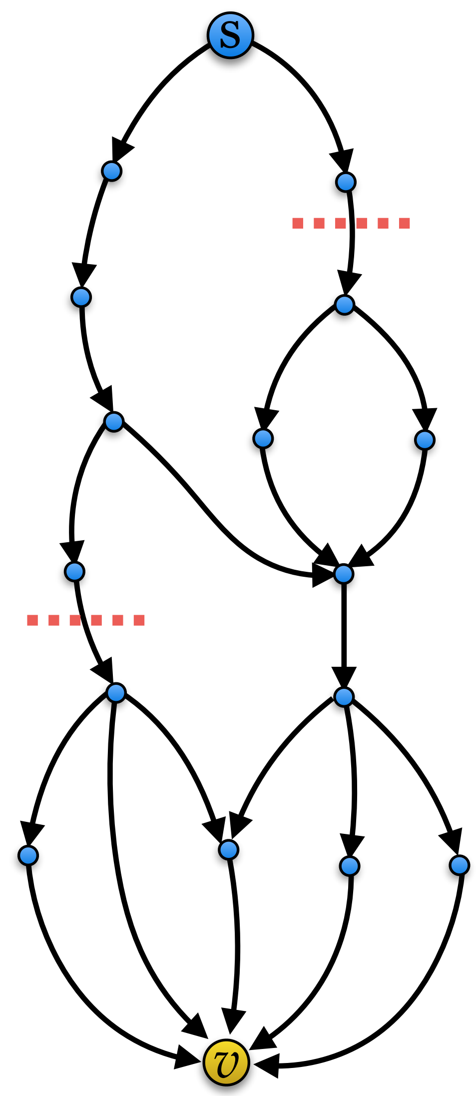
Farthest min-cut



Problem 1: Reachability Preserver

Proof Snippet

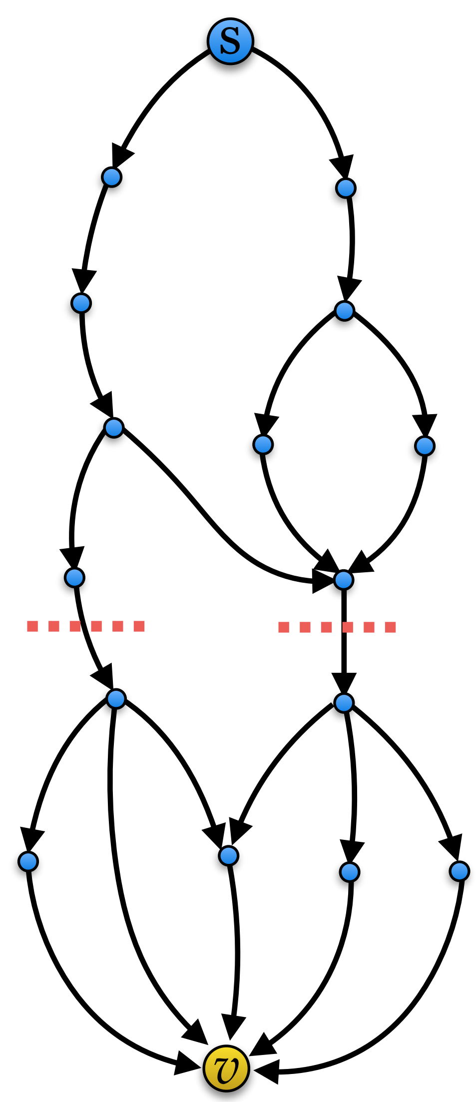
Farthest min-cut



Problem 1: Reachability Preserver

Proof Snippet

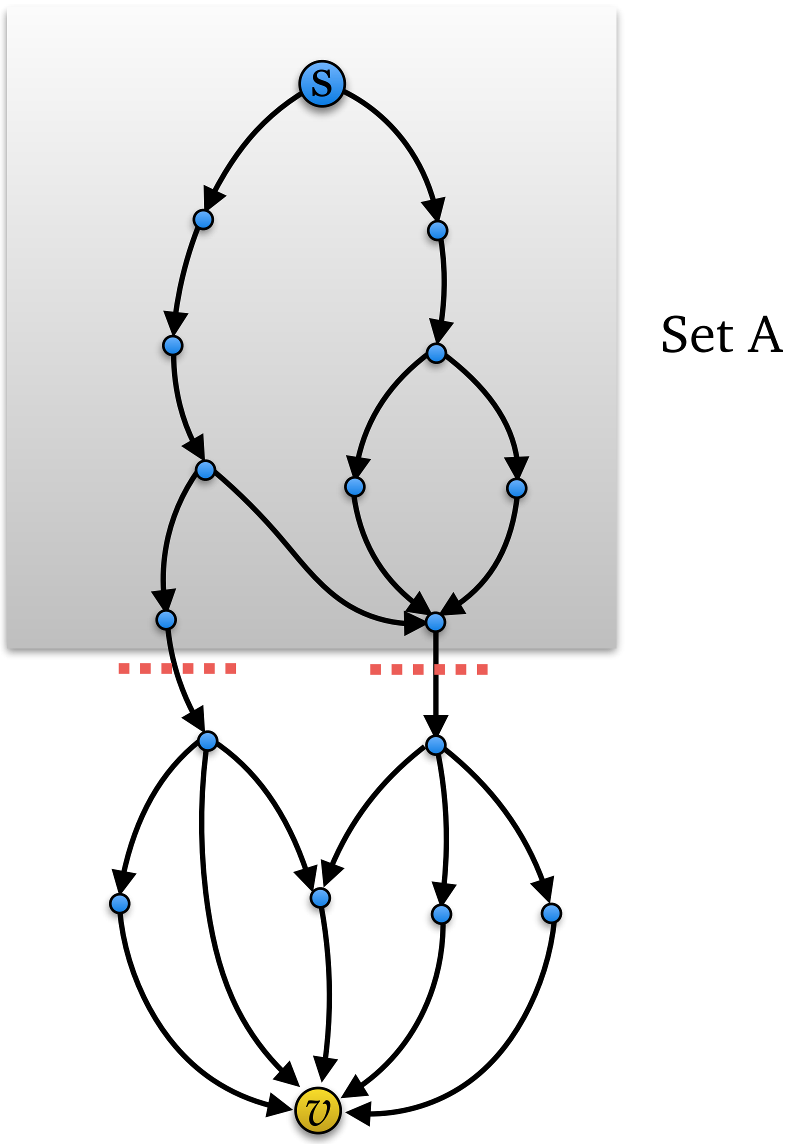
Farthest min-cut



Problem 1: Reachability Preserver

Proof Snippet

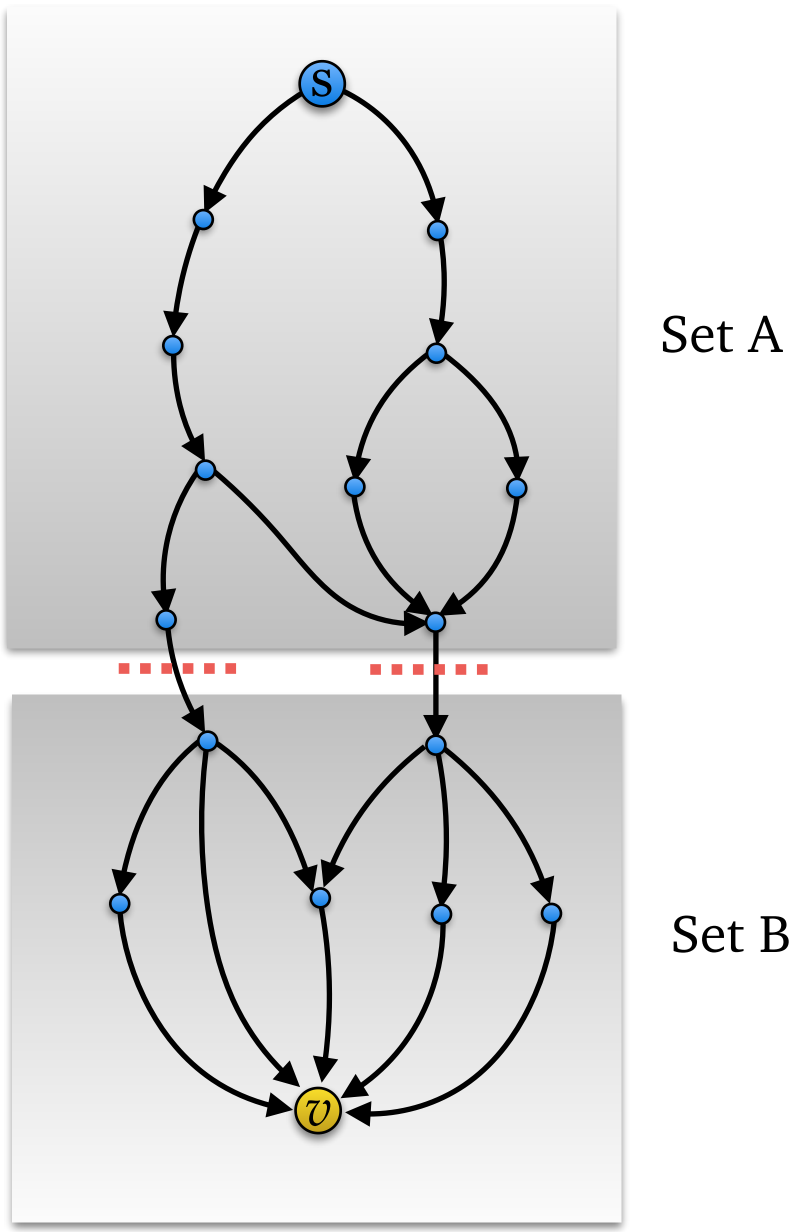
Farthest min-cut



Problem 1: Reachability Preserver

Proof Snippet

Farthest min-cut



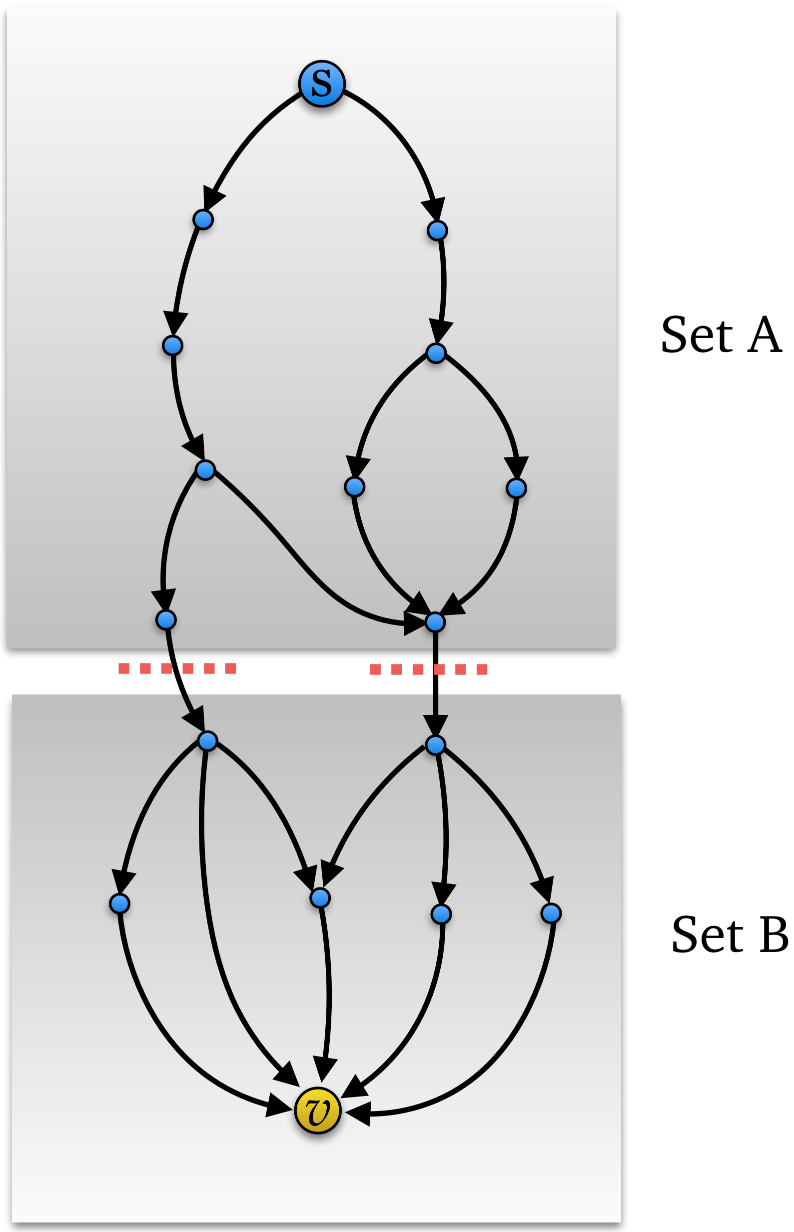
Problem 1: Reachability Preserver

Proof Snippet

Farthest min-cut

Definition

The min cut $\{A,B\}$ for which the set A **is of maximum size**.



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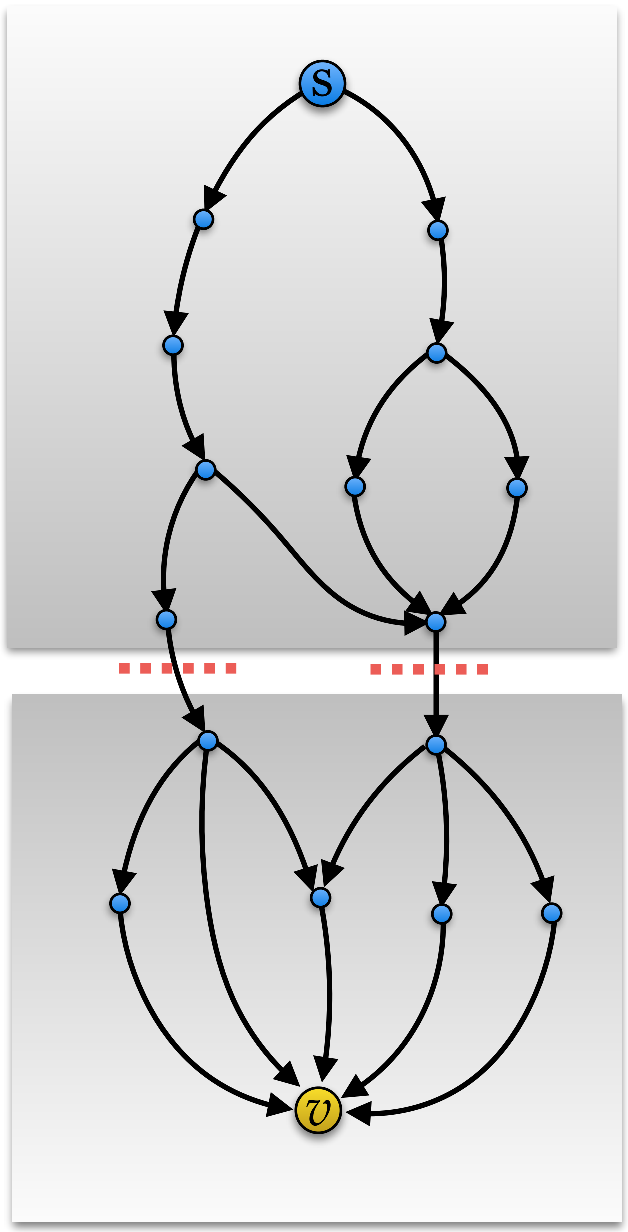

Set A

Farthest (s,v)-min-cut

Set B

Problem 1: Reachability Preserver

Proof Snippet

Farthest min-cut

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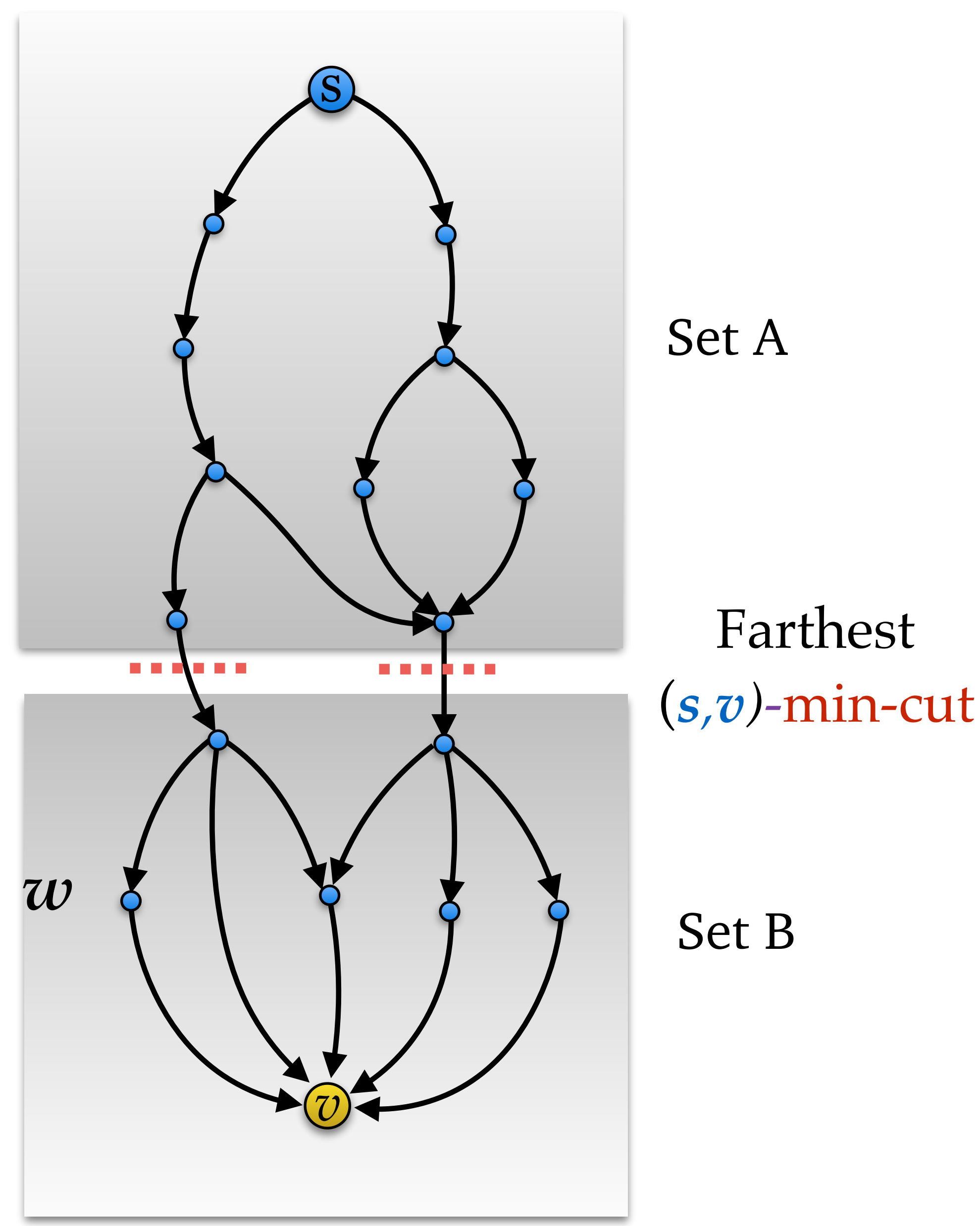


Characterisation

Vertex w lies in B, iff

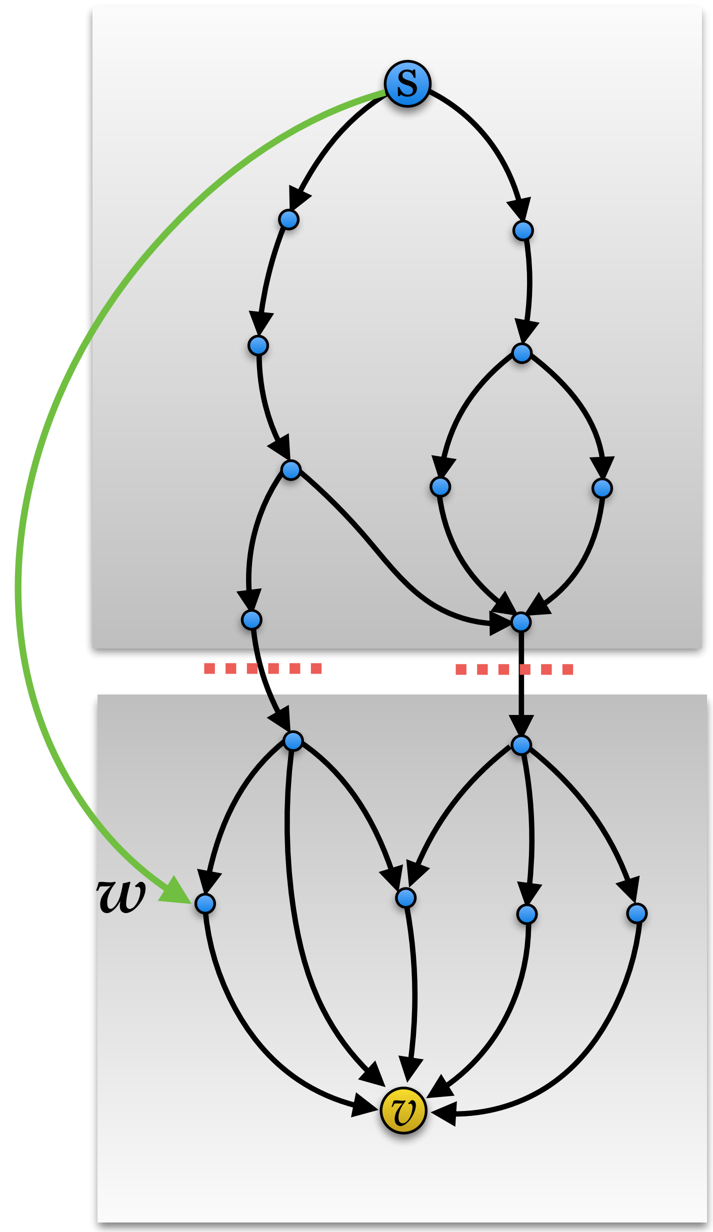
Problem 1: Reachability Preserver

Proof Snippet

Farthest min-cut

Definition

The min cut $\{A,B\}$ for which the set A **is of maximum size**.



Set A

Farthest (s,v)-min-cut

Set B

Characterisation

Vertex w lies in B, iff

Problem 1: Reachability Preserver

Proof Snippet

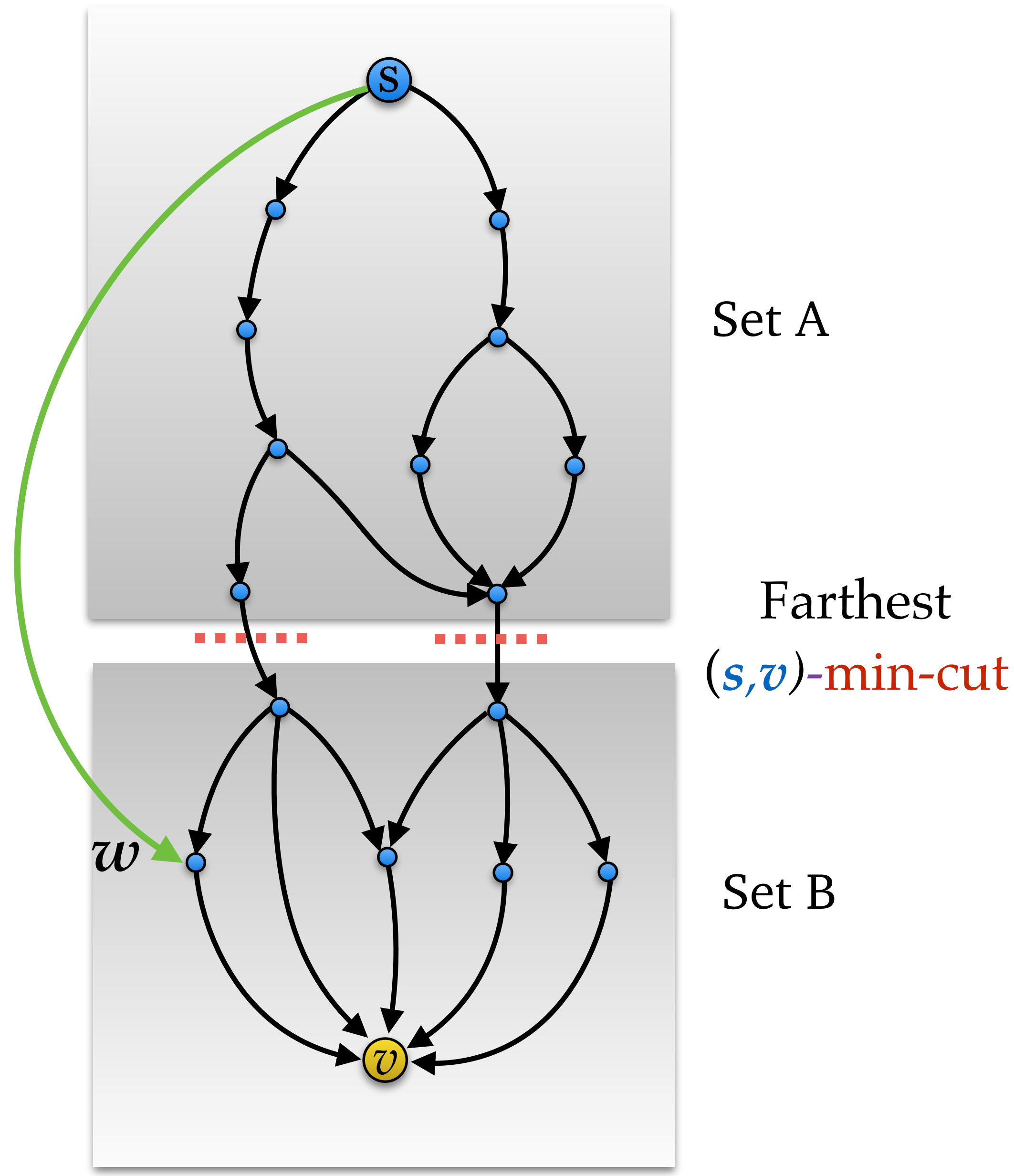
Farthest min-cut

Definition

The min cut $\{A,B\}$ for which the set A **is of maximum size**.

Characterisation

Vertex w lies in B, iff $\textit{max-flow}(G+(s,w)) > \textit{max-flow}(G)$



Problem 1: Reachability Preserver

Proof Snippet

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: $(k+1)!$

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s **x** v - preserver , with bounded in-degree(v)

*s **x** $V(G)$ - preserver*

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s **x** v - preserver , with bounded in-degree(v)

s* x *V(G)* - *preserver

Preserves reachability from s to **each** vertex of G

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: $(k+1)$!

Bottleneck: s x v - preserver , with bounded in-degree(v)

s* x *V(G)* - *preserver

Preserves reachability from s to **each** vertex of G

upon failure of at most k edges

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: $(k+1)$!

Bottleneck: s x v - preserver , with bounded in-degree(v)

s* x *V(G)* - *preserver

Preserves reachability from s to **each** vertex of G

upon failure of at most k edges

Main Goal

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s x v - preserver , with bounded in-degree(v)

s **x** *V(G)* - *preserver*

Preserves reachability from s to **each** vertex of G

upon failure of at most k edges

Main Goal

s **x** *V(G)* - *preserver* in which in-degree of **each vertex** is bounded

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: $(k+1)$!

Bottleneck: s x v - preserver , with bounded in-degree(v)

s* x *V(G)* - *preserver

Preserves reachability from s to **each** vertex of G upon failure of at most k edges

Main Goal

s **x** *V(G)* - *preserver* in which in-degree of **each vertex** is bounded

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: $(k+1)$!

Bottleneck: s x v - preserver , with bounded in-degree(v)

s* x *V(G)* - *preserver

Preserves reachability from s to **each** vertex of G upon failure of at most k edges

Main Goal

s **x** *V(G)* - *preserver* in which in-degree of **each vertex** is bounded

s* x *v* - *preserver

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s **x** v - preserver , with bounded in-degree(v)

<u>*s* **x** *V(G) - preserver*</u>

Preserves reachability from s to **<u>each</u>** vertex of G

upon failure of at most k edges

Main Goal

s **x** *V(G) - preserver* in which in-degree of **<u>each vertex</u>** is bounded

<u>*s* **x** *v - preserver*</u>

Preserves reachability from s to **<u>only</u>** v

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

s* x *V(G)* - *preserver

Preserves reachability from s to **each** vertex of G

upon failure of at most k edges

Main Goal

$s \times V(G)$ - *preserver* in which in-degree of **each vertex** is bounded

s* x *v* - *preserver

Preserves reachability from s to **only** v

upon failure of at most k edges

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

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Preserves reachability from s to **each** vertex of G upon failure of at most k edges

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Main Goal

s **x** *V(G)* - *preserver* in which in-degree of **each vertex** is bounded

A Simpler Problem

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s x v - preserver , with bounded in-degree(v)

s **x** *V(G) - preserver*

Preserves reachability from s to **each** vertex of G upon failure of at most k edges

Main Goal

s **x** *V(G) - preserver* in which in-degree of **each vertex** is bounded

s **x** *v - preserver*

Preserves reachability from s to **only** v upon failure of at most k edges

A Simpler Problem

s **x** *v - preserver* in which in-degree of **only** v is bounded

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s x v - preserver , with bounded in-degree(v)

***s* x *V(G)* - preserver**

Preserves reachability from s to **each** vertex of G upon failure of at most k edges

***s* x *v* - preserver**

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Main Goal

s x $V(G)$ - *preserver* in which in-degree of **each vertex** is bounded

A Simpler Problem

s x v - *preserver* in which in-degree of **only** v is bounded

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s x v - preserver , with bounded in-degree(v)

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s x v - preserver , with bounded in-degree(v)

Case 1: Max-Flow(s,v) ≥ k+1

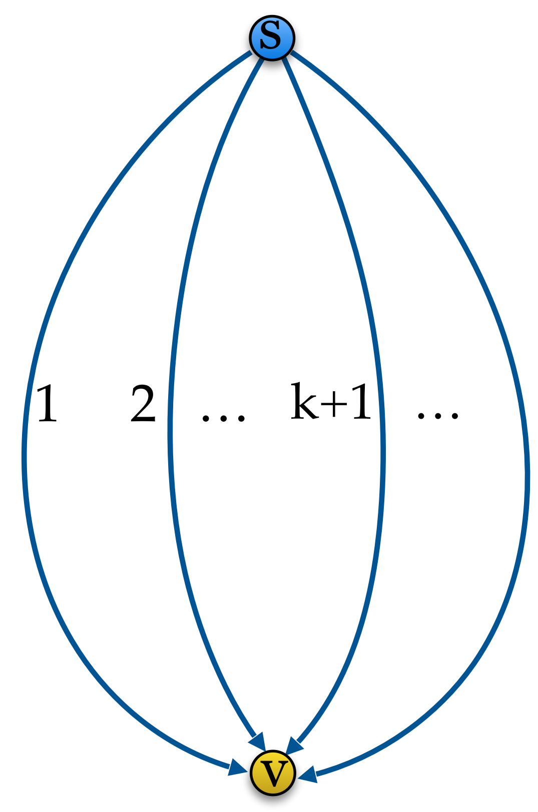
Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

Case 1: Max-Flow(s,v) ≥ k+1



Problem 1: Reachability Preserver

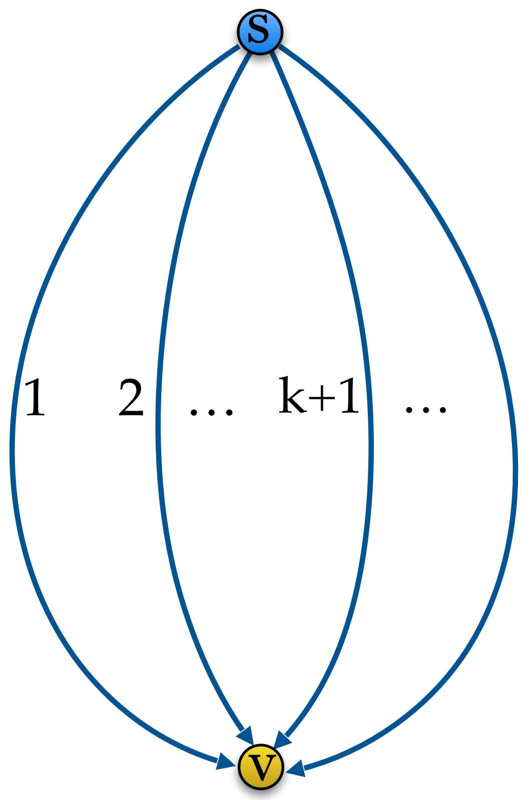
Proof Snippet

In-degree at most: ***(k+1)***!

Bottleneck: *s* x *v* - preserver , with bounded in-degree(v)

Case 1: Max-Flow(s,v) ≥ k+1

H = Union of any ***k+1*** edge-disjoint paths



Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: s x v - preserver , with bounded in-degree(v)

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

Case 2: Max-Flow(s,v) = r < k+1

Problem 1: Reachability Preserver

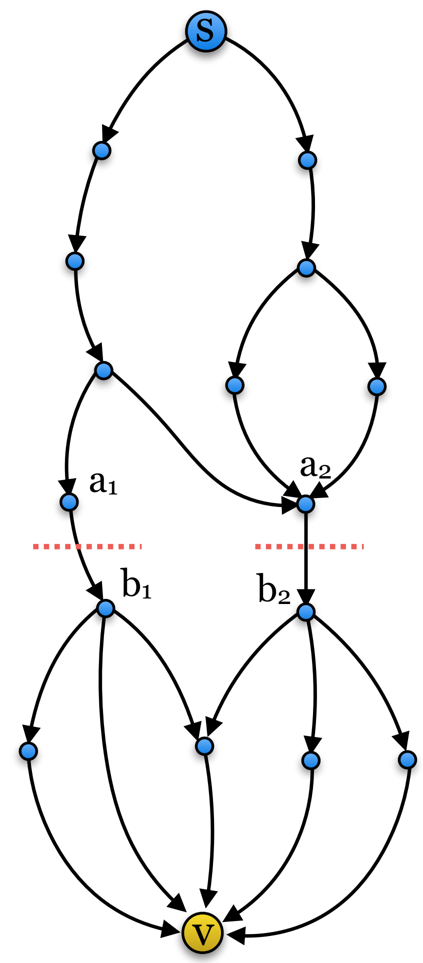
Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

Case 2: Max-Flow(s,v) = r < k+1

- Let farthest Min-cut = $\{(a_1,b_1), \ldots, (a_r,b_r)\}$



Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

Case 2: Max-Flow(s,v) = r < k+1

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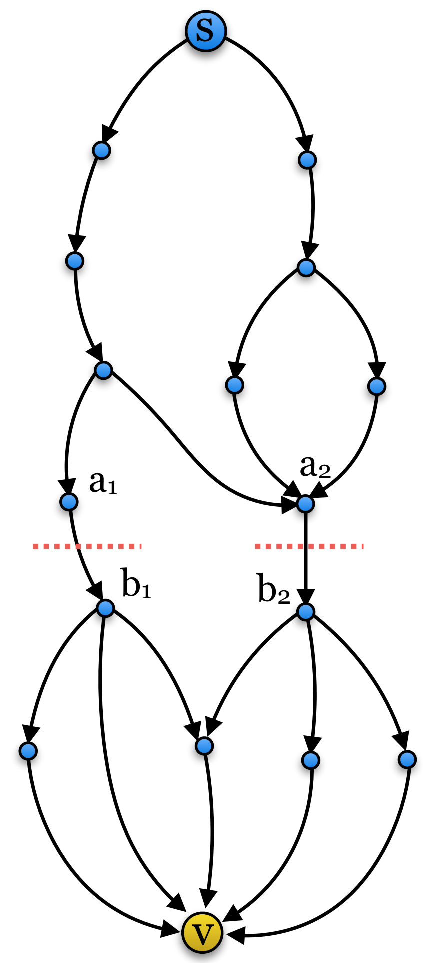

$G_1 := G+(s,b_1)$

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

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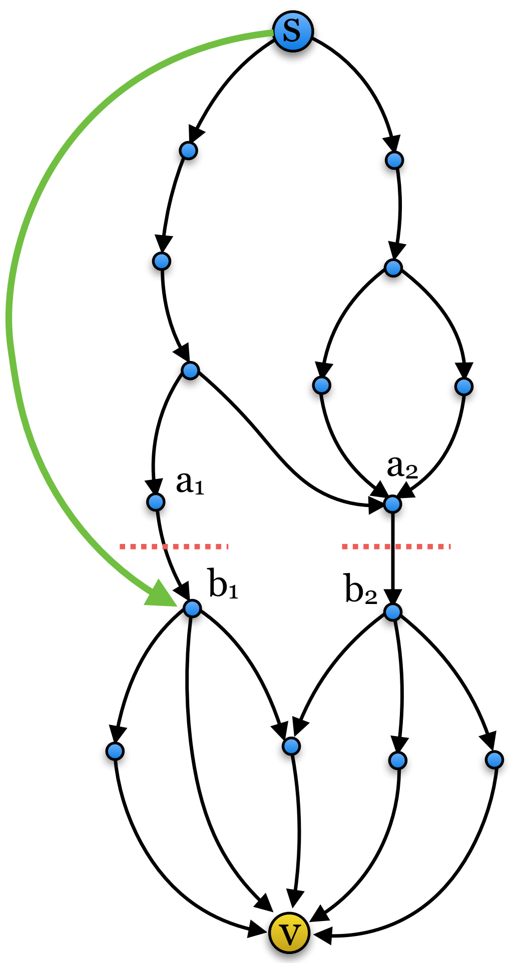

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Proof Snippet

In-degree at most: *(k+1)*!

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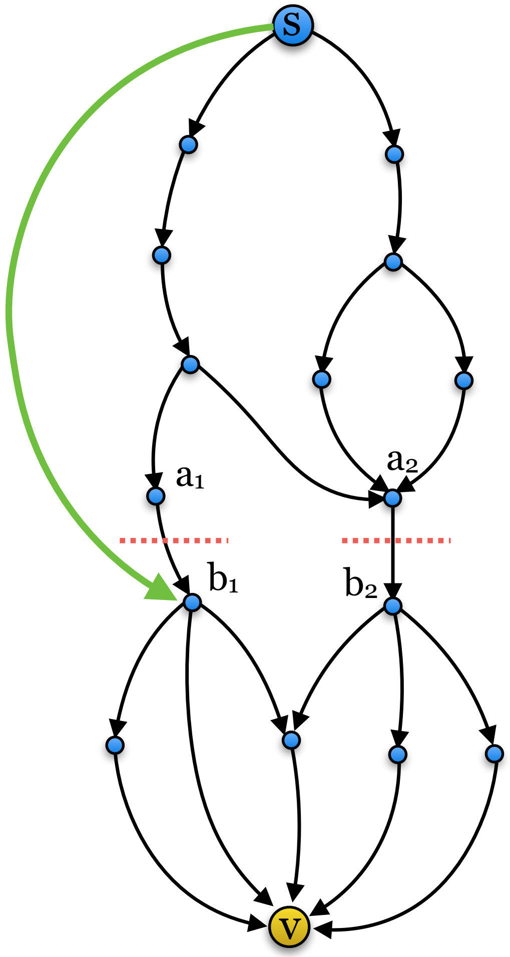
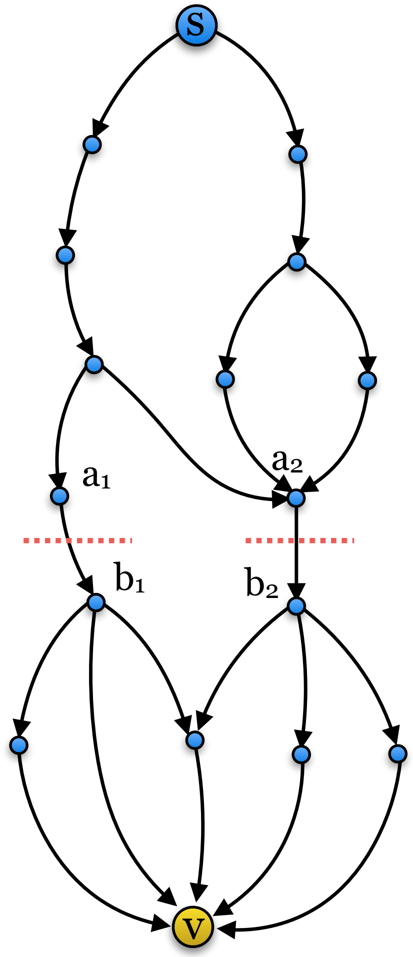


$G_1 := G+(s,b_1)$



Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

Case 2: Max-Flow(s,v) = r < k+1

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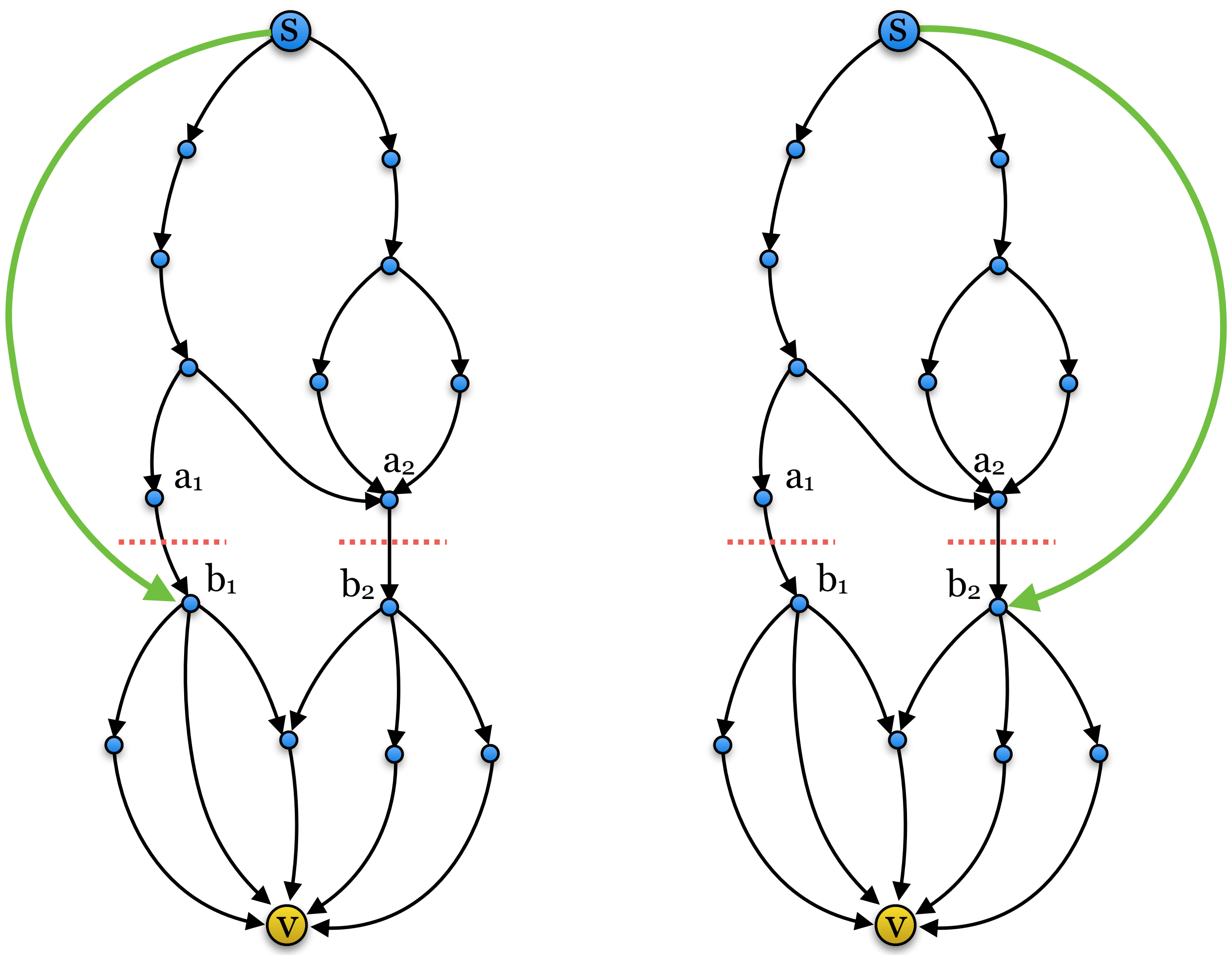

$G_1 := G+(s,b_1)$

Problem 1: Reachability Preserver

Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver, with bounded in-degree(v)

Case 2: Max-Flow(s,v) = r < k+1

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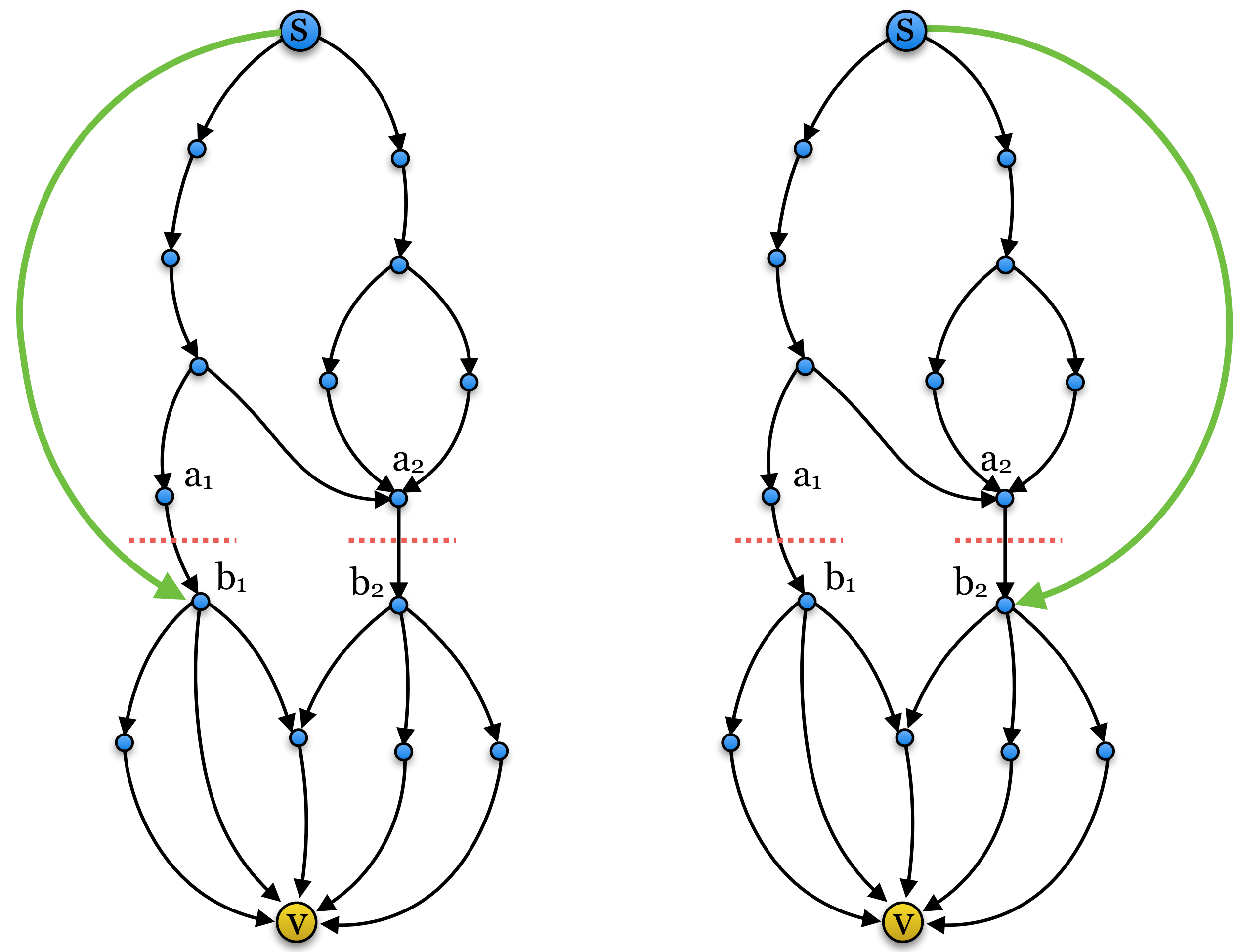


$G_1 := G+(s,b_1)$ $\quad$ $G_2 := G+(s,b_2)$

Problem 1: Reachability Preserver

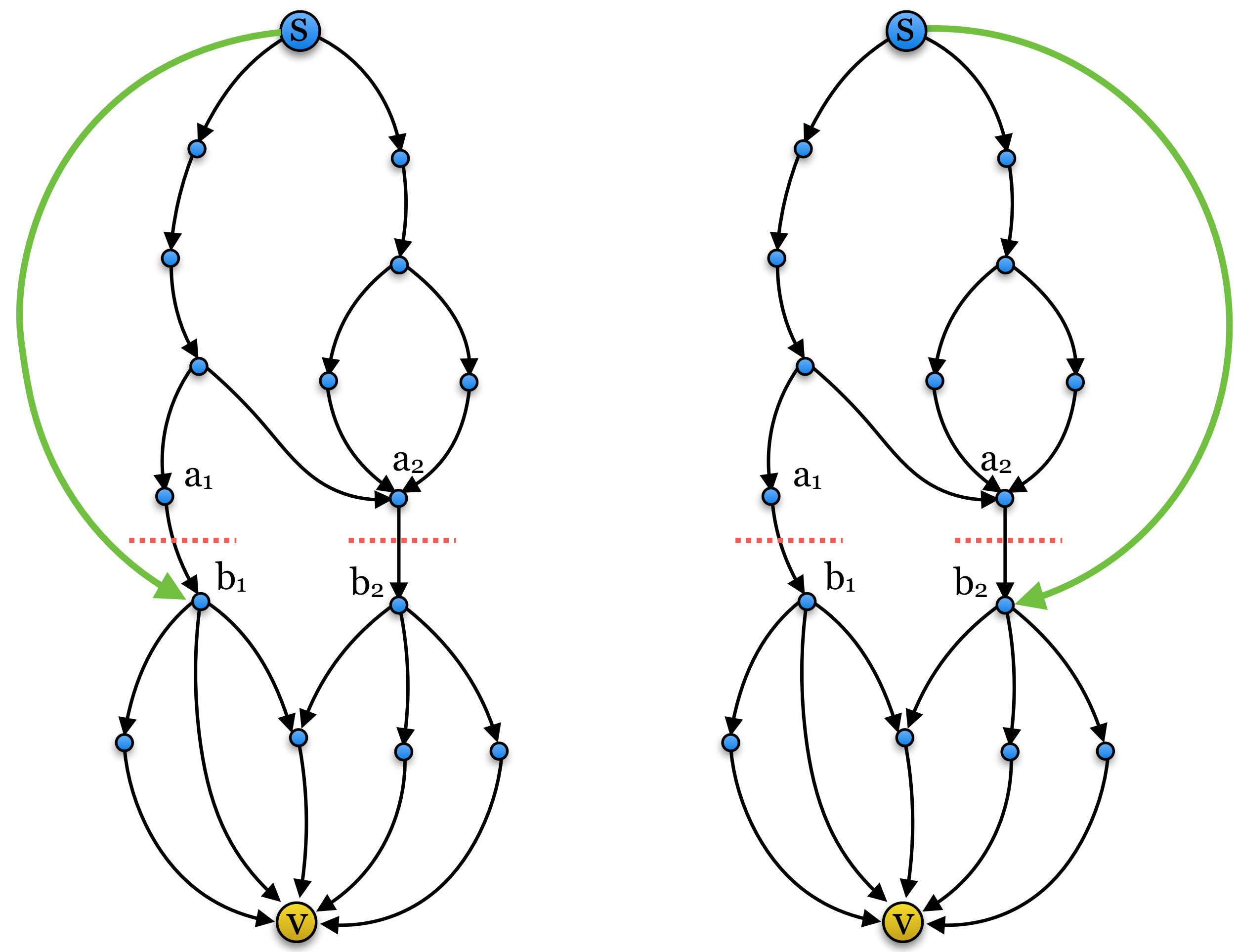
Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

Case 2: Max-Flow(s,v) = r < k+1

- Let farthest Min-cut = $\{(a_1,b_1), \ldots, (a_r,b_r)\}$
- Find Preserver (say H_i) w.r.t. $G_i = G+(s,b_i)$



$G_1 := G+(s,b_1)$ $G_2 := G+(s,b_2)$

Problem 1: Reachability Preserver

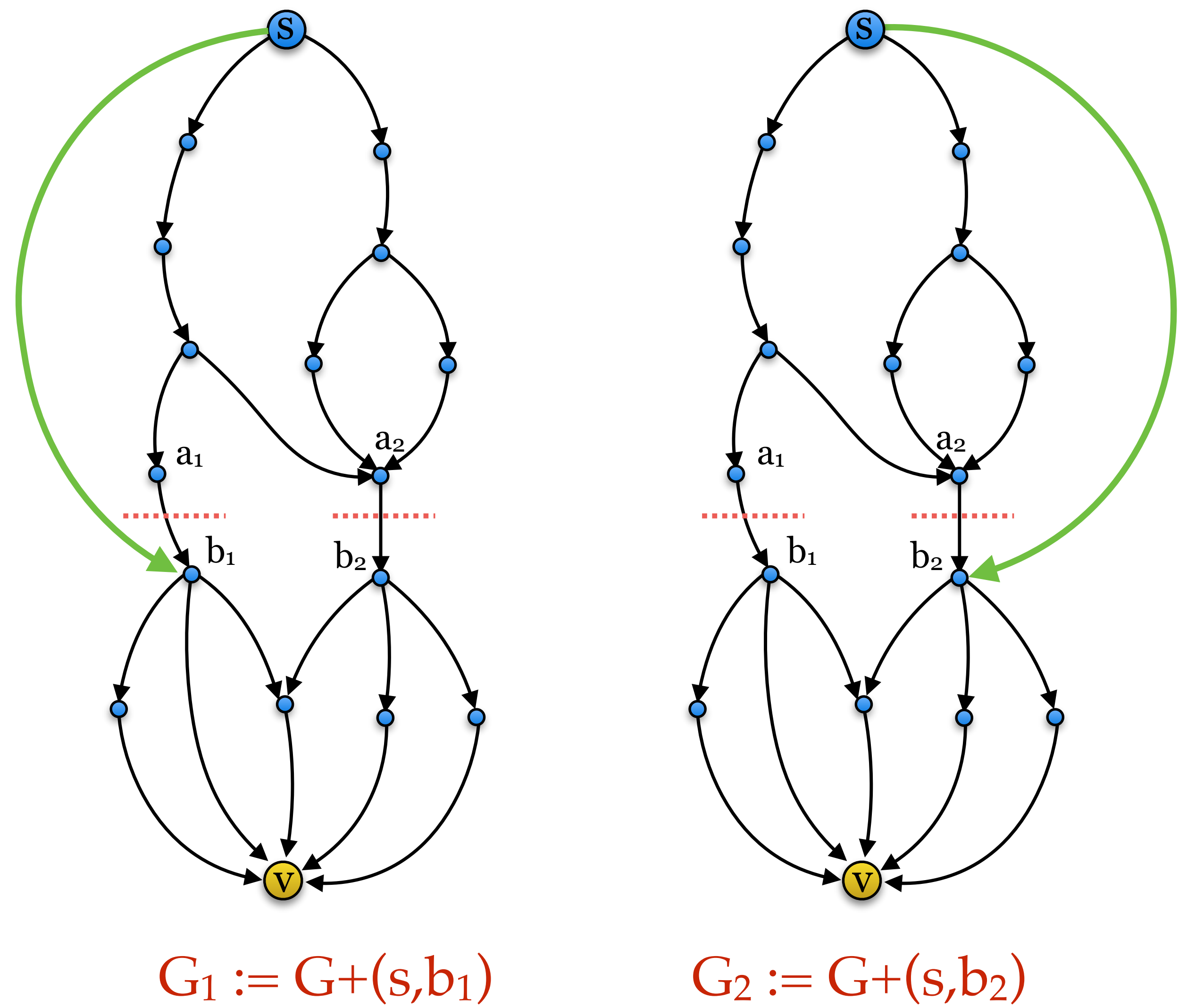
Proof Snippet

In-degree at most: *(k+1)*!

Bottleneck: $s \times v$ - preserver , with bounded in-degree(v)

Case 2: Max-Flow(s,v) = r < k+1

- Let farthest Min-cut = $\{(a_1,b_1), \ldots, (a_r,b_r)\}$
- Find Preserver (say H_i) w.r.t. $G_i = G+(s,b_i)$
- SET: $E(v,H) = \bigcup_{i=1 \text{ to } r} E(v,H_i)$



Problem 2: SCC Oracle

Input: directed graph $G=(V,E)$, parameter k.

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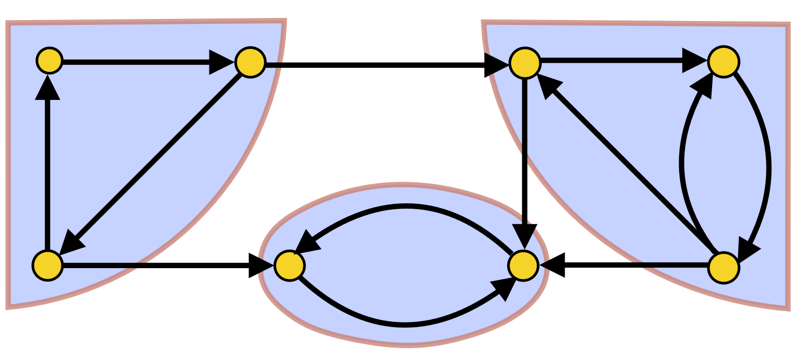

G

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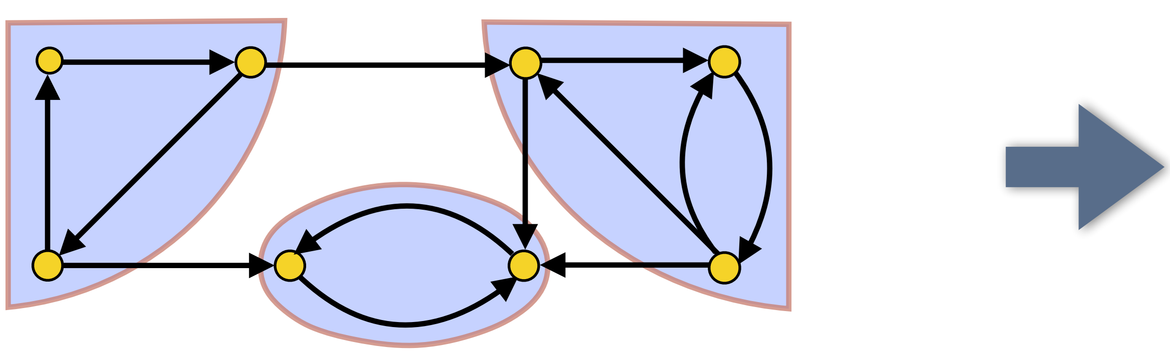

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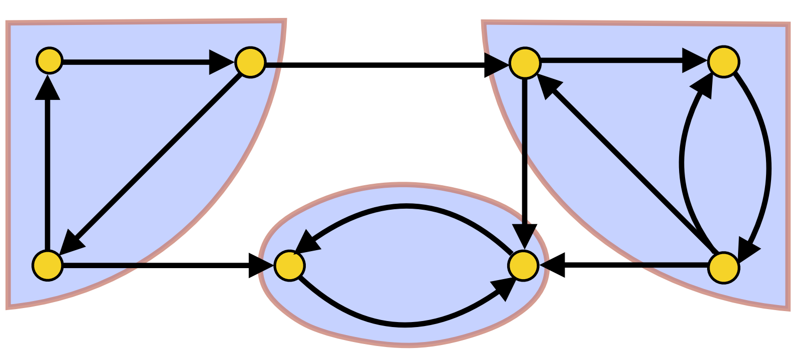

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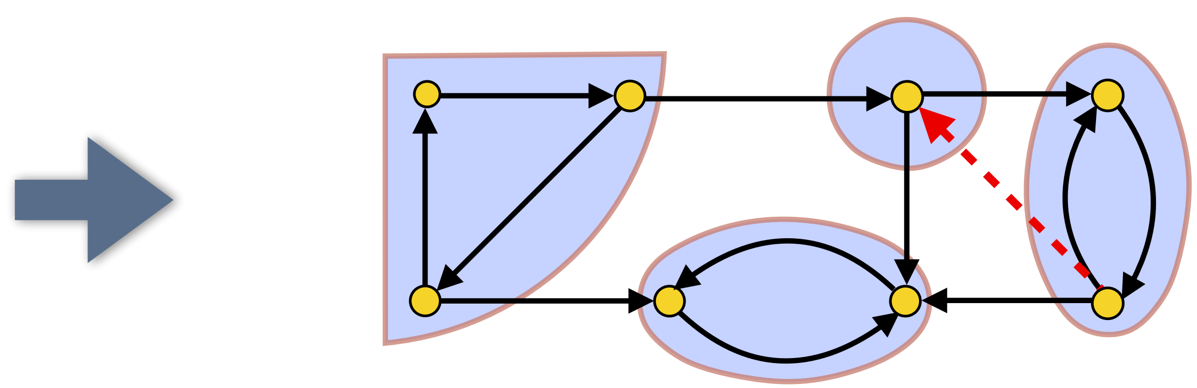

SCCs in G\F

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Prior Work:

[Italiano et al. (2017)]:

- k=1 (single failure)
- An oracle of $O(n)$ *size*
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Proof Snippet

Bottleneck: SCCs intersecting fixed path P

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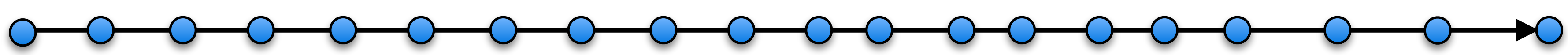
Lemma:

If we can compute **SCCs in G\F intersecting a path "P" in $F(n,k)$ time**, then, we can compute ***ALL* the SCCs of G\F in $O(F(n,k) \log n)$** time.

Problem 2: SCC Oracle

Proof Snippet

Bottleneck: SCCs intersecting fixed path P



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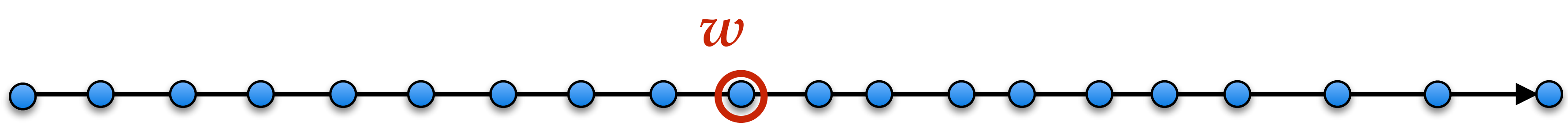

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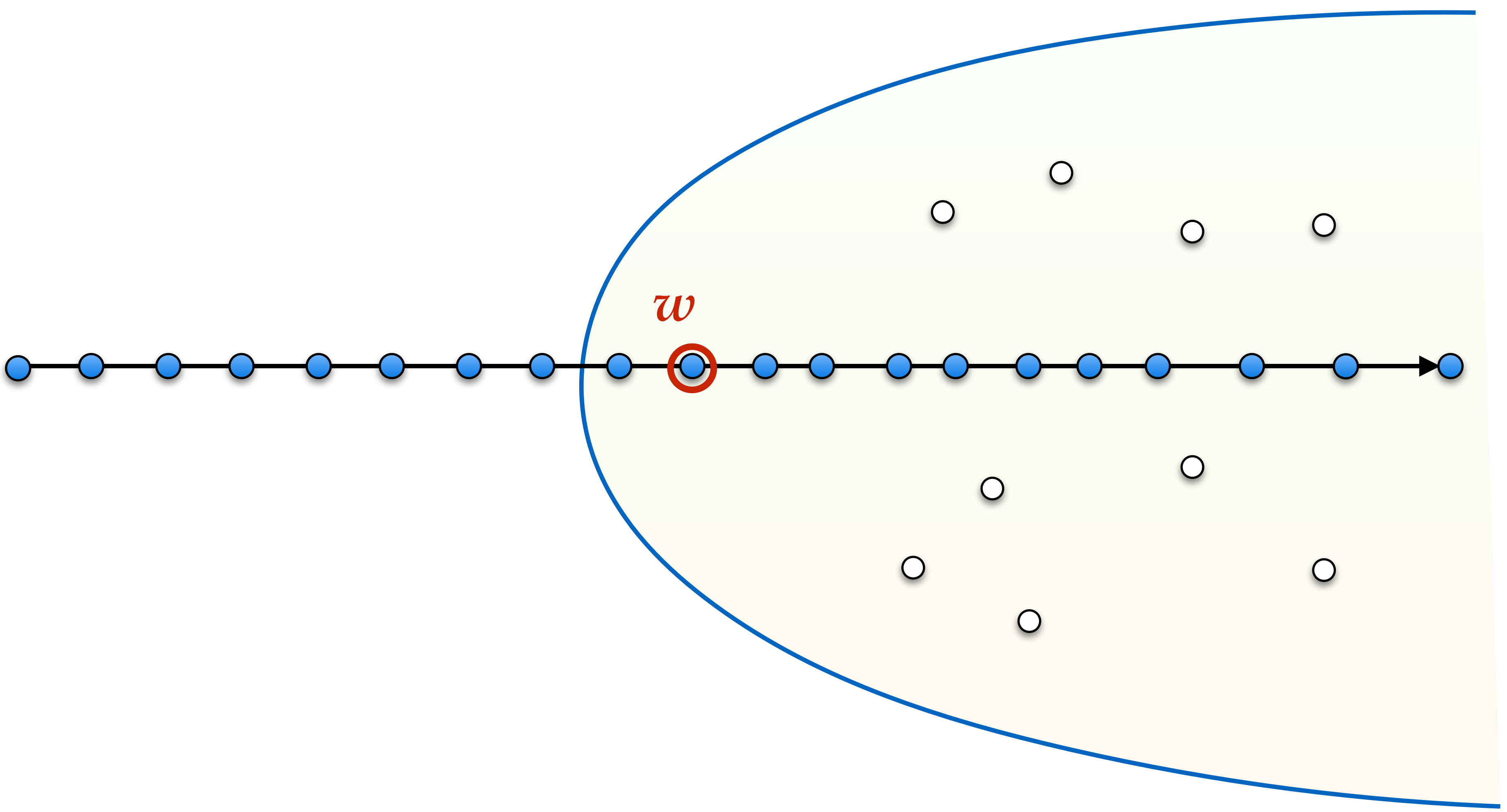

$V_1 =$ vertices reachable from w in $G \setminus F$

Problem 2: SCC Oracle

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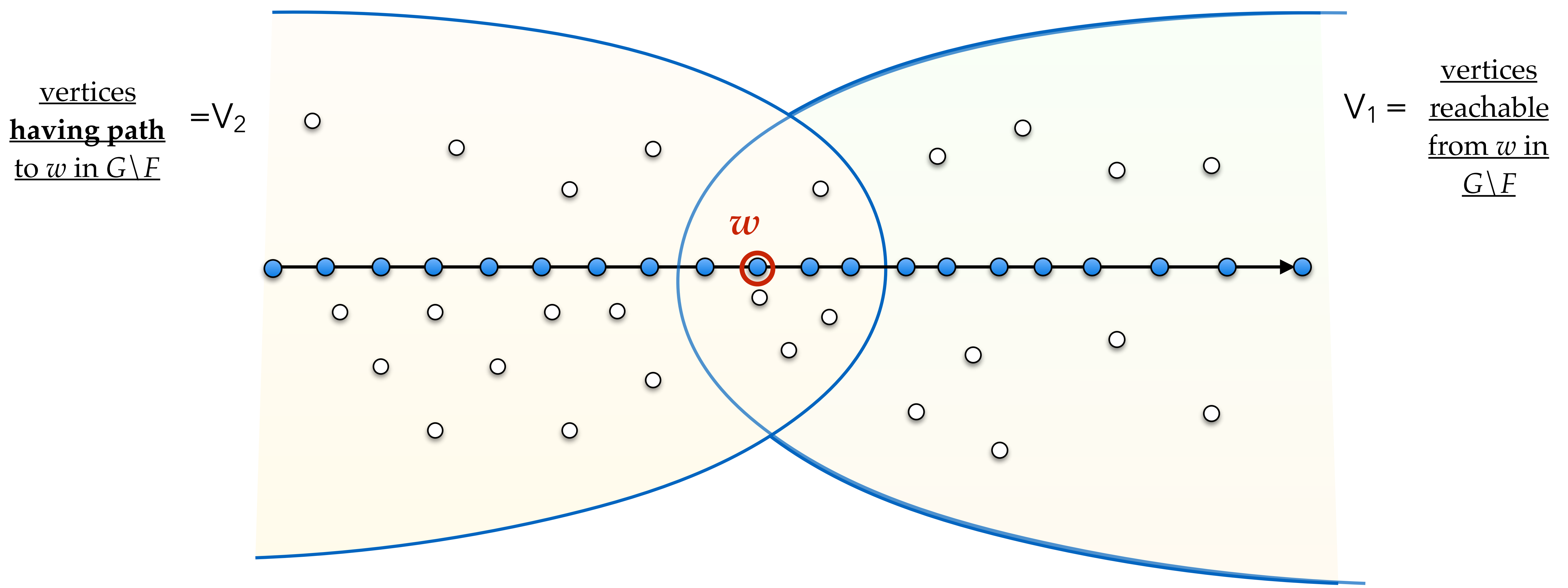

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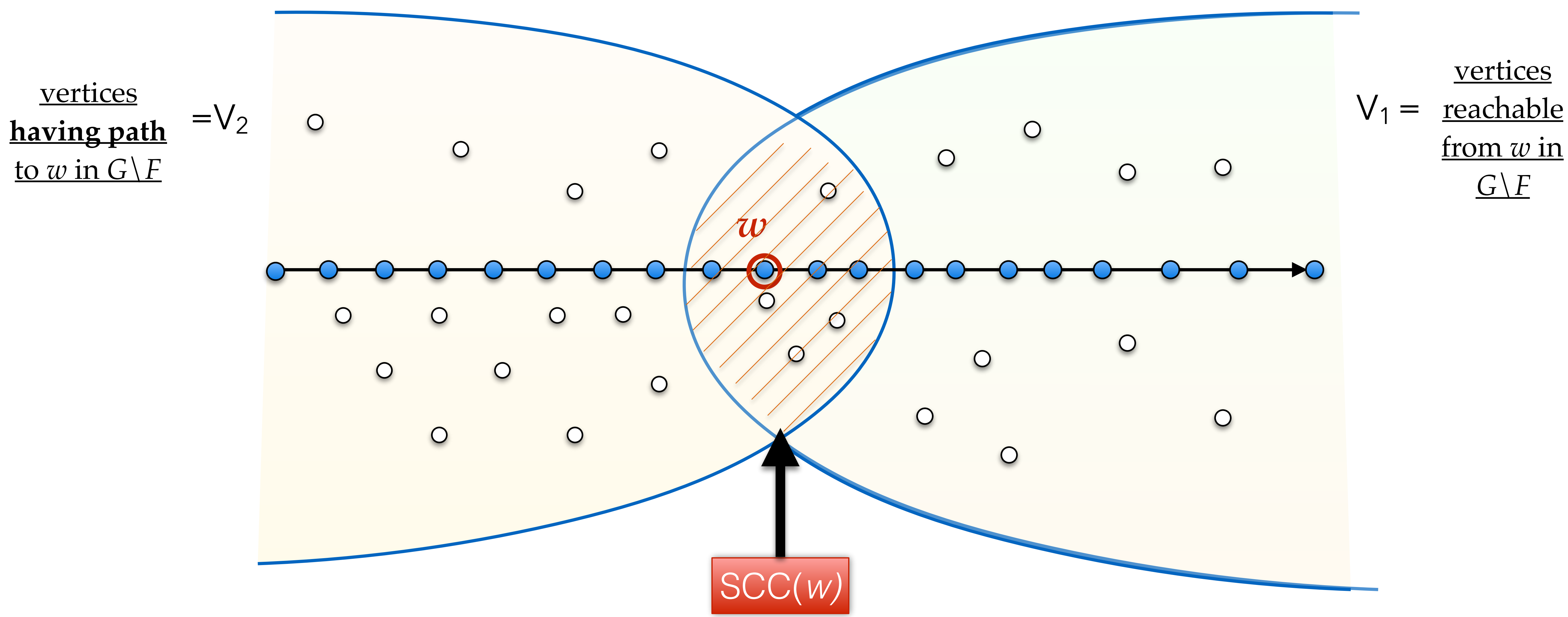

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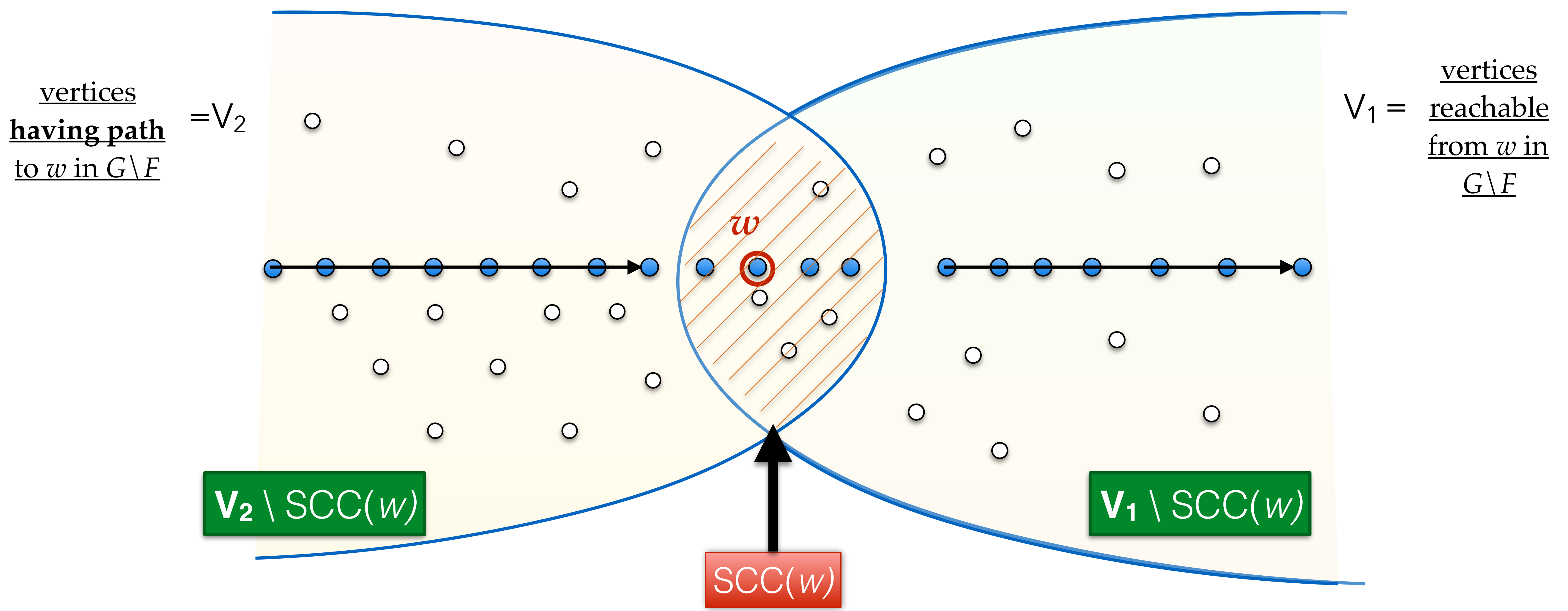

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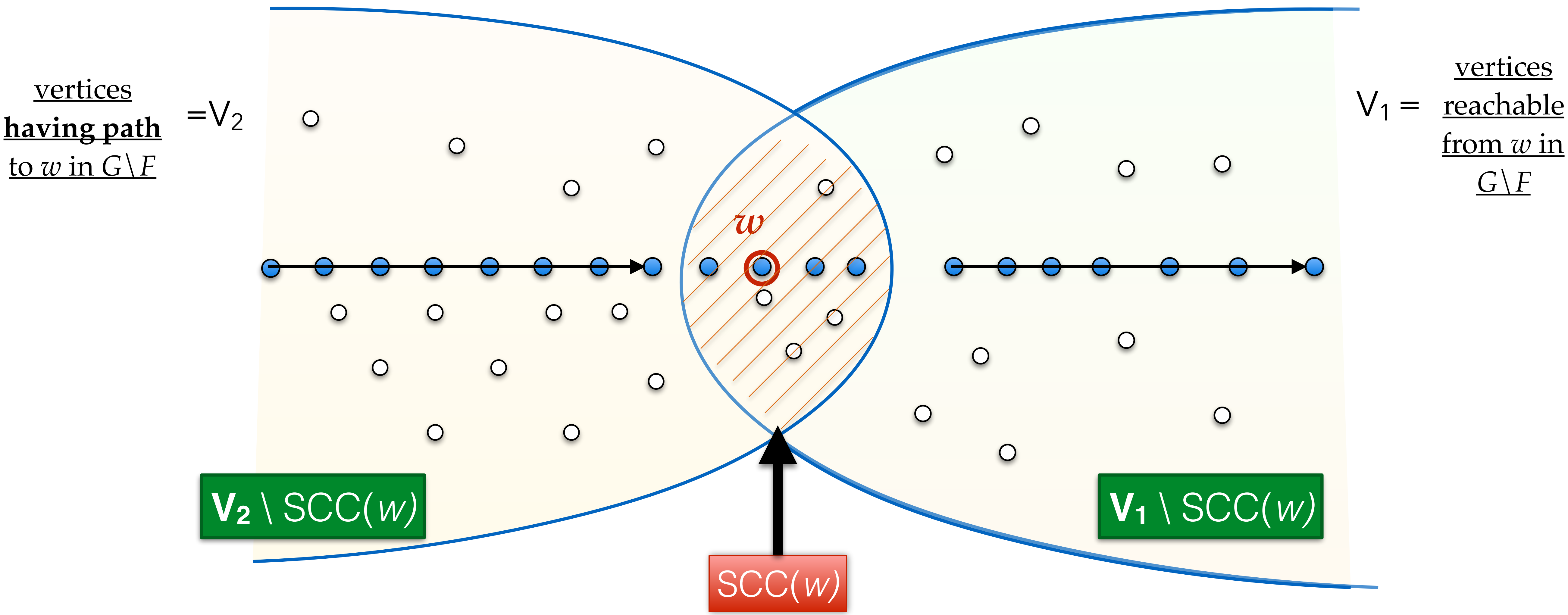

In $O(2^k n)$ time — divide problem into two sub-problems

Problem 2: SCC Oracle

Proof Snippet

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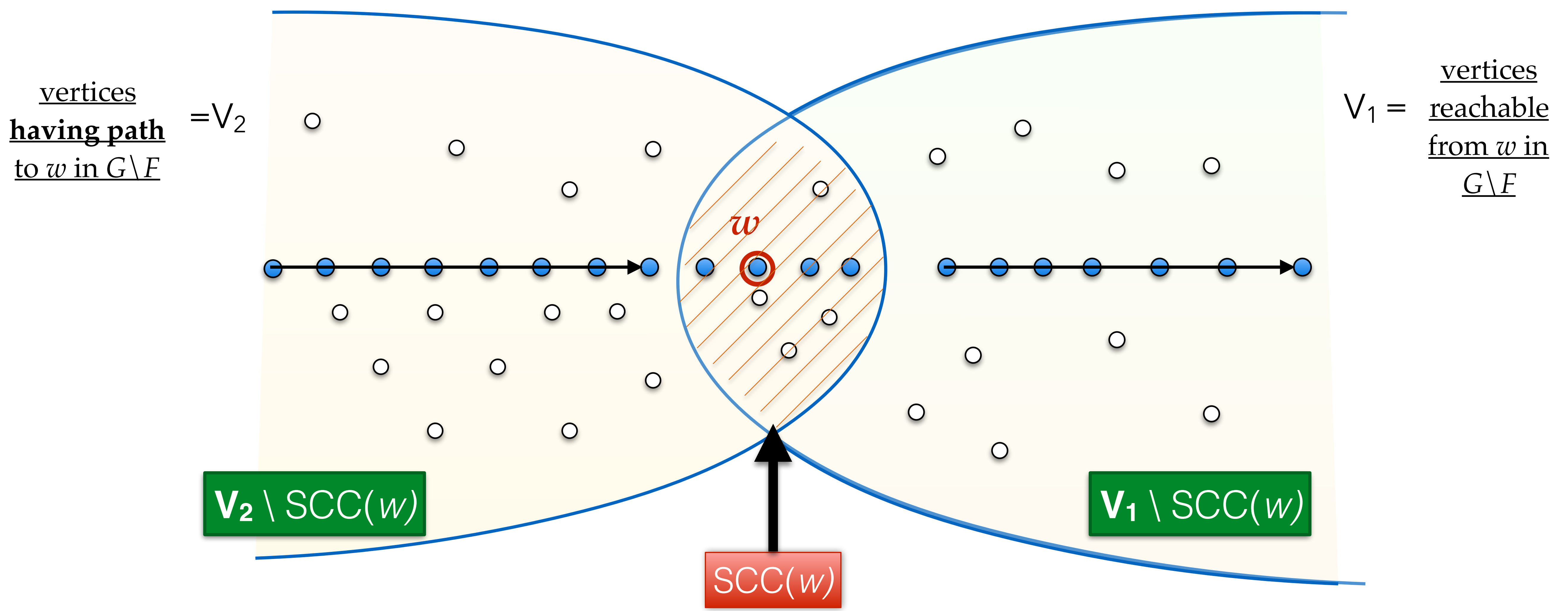

In $O(2^k n)$ time — divide problem into two sub-problems

Recursively solve in $O(2^k n \log |P|)$ time

Problem 2: SCC Oracle

Proof Snippet

Computing all SCCs

Lemma:

If we can compute **SCCs in G\F intersecting a path "P" in $F(n,k)$ time**, then, we can compute ***ALL* the SCCs of G\F in $O(F(n,k)\log n)$** time.

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If we can compute **SCCs in G\F intersecting a path "P" in $F(n,k)$ time**, then, we can compute ***ALL* the SCCs of G\F in O($F(n,k)$) log n)** time.

<u>**Main Result:**</u>

For any set F of k failures, we can compute SCCs of graph $G\backslash F$ in
<u>$O(2^k n \log^2 n)$</u> time.

Problem 2: SCC Oracle

Proof Snippet

Computing all SCCs

Lemma:

If we can compute **SCCs in G\F intersecting a path "P" in $F(n,k)$ time**, then, we can compute ***ALL*** **the SCCs of G\F in $O(F(n,k) \log n)$** time.

Main Result:

For any set F of k failures, we can compute SCCs of graph $G\backslash F$ in $\underline{O(2^k n \log^2 n)}$ time.

Size of the oracle is $O(2^k n^2)$.

Thank You